Day 4

- A Gem of Combinatorics
組合わせ論の宝石
  - Dilworth’s theorem
  - Matching –Covering duality

Sperner system

- Given a set S of size n, a family **F** of subsets S is called a Sperner system if no pair in **F** is in the inclusion relation, that is, there are no pair A, B in **F** such that A ⊂B

互いに包含関係にない集合の族

- Example, the set of all subsets of cardinality k is a Sperner system.
- Problem: Show that the largest Sperner system has cardinality $_{n}C_{[n/2]}$
- スペルナー系の最大の大きさは$_{n}C_{[n/2]}$であることを示せ

Proof by counting

- Chain of sets: sequence of sets with inclusion relation

チェイン： 包含関係にある集合列

- Given a Sperner system **F**, count the number of pairs (A, *C*) such that A is in **F** and *C* is a chain containing A.

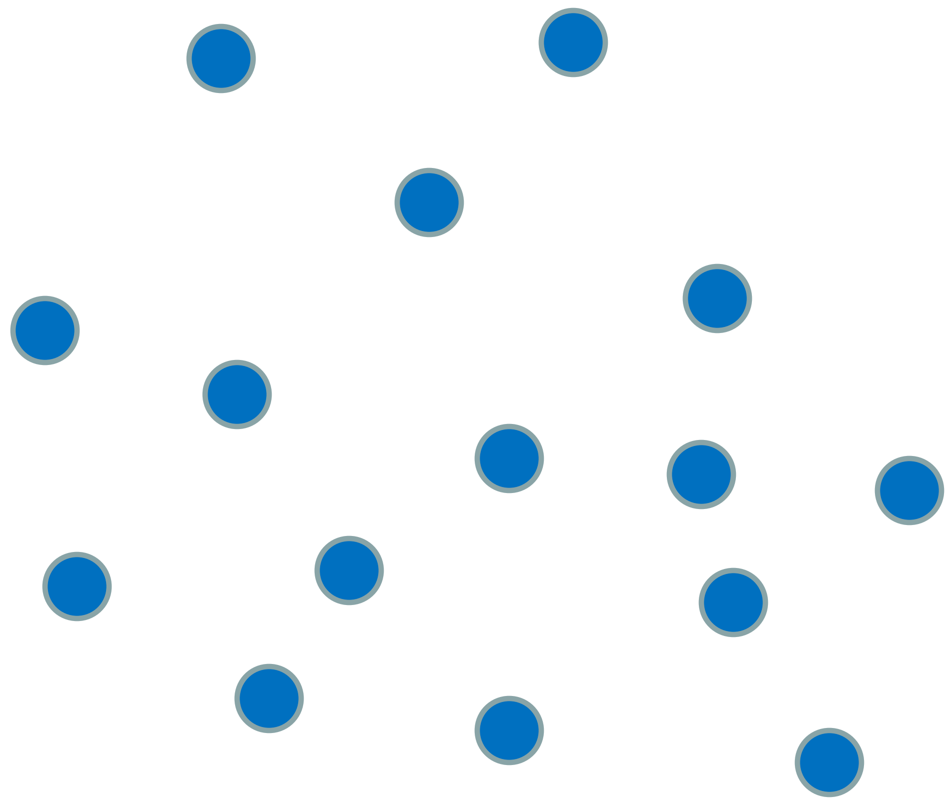
Fに入るAとAを含むチェインCの対（A,C)を数える

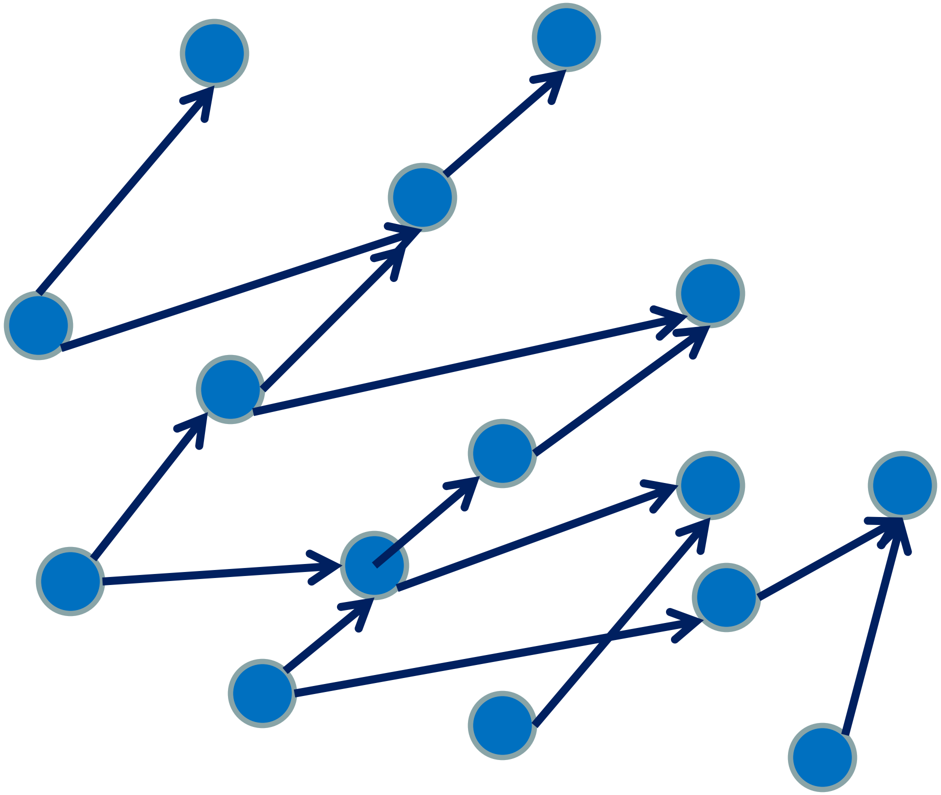
Gem of Combinatorics

Chain, Antichain, Matching, Covering, Independent Sets, and combinatorial dualities

Partially Ordered Set (poset)

- Given a set S, a relation > is called a partial order if it satisfies the follwoing axioms
  - If x > y and y> z then x>z
  - x >y and y<x if and only if x=y
- A set with a partial order is called a poset
  - Example 1. A set of numbers
  - Example 2. A set of points (in which order?)
  - Example 3. A set of subsets of a finite set S

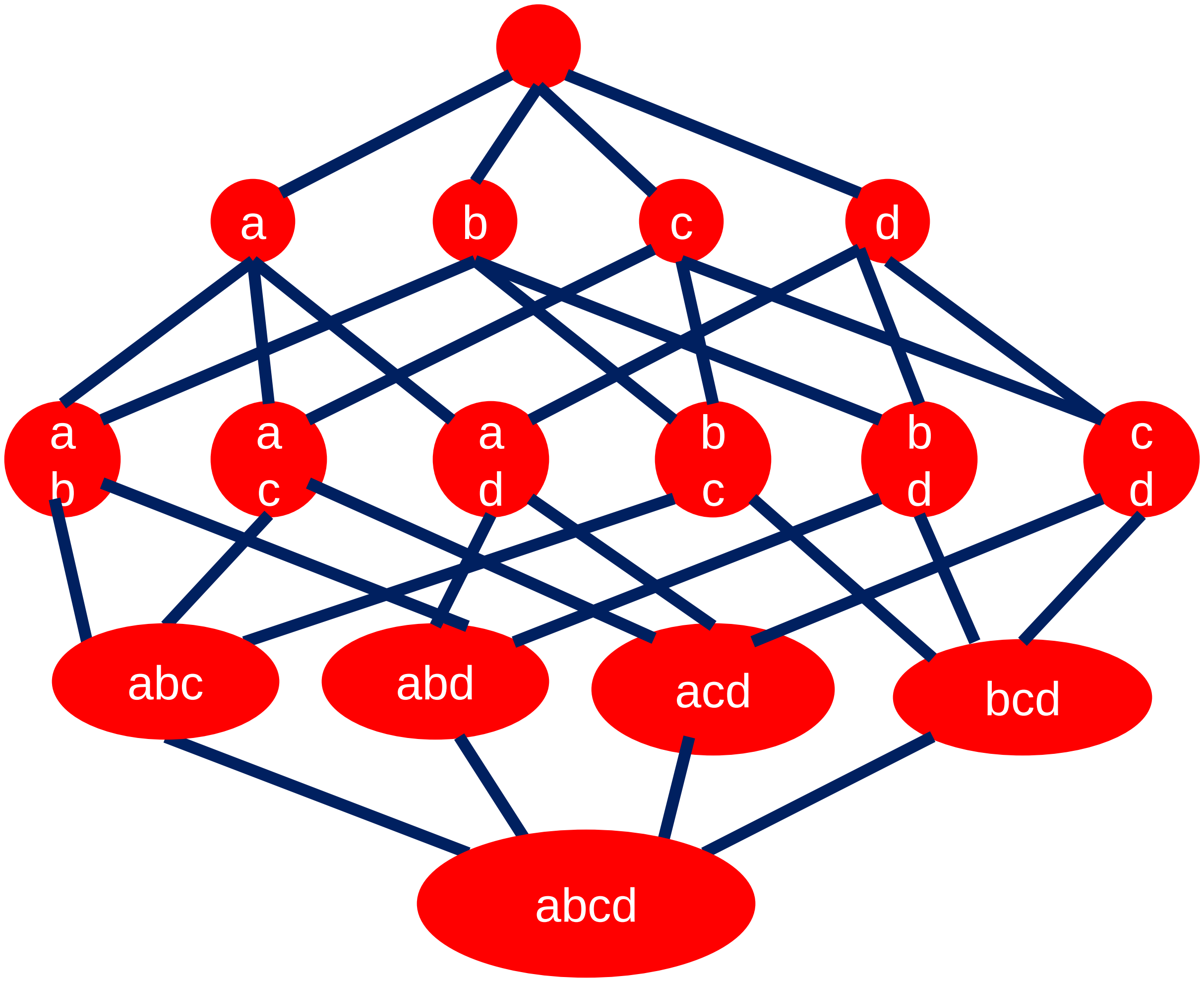




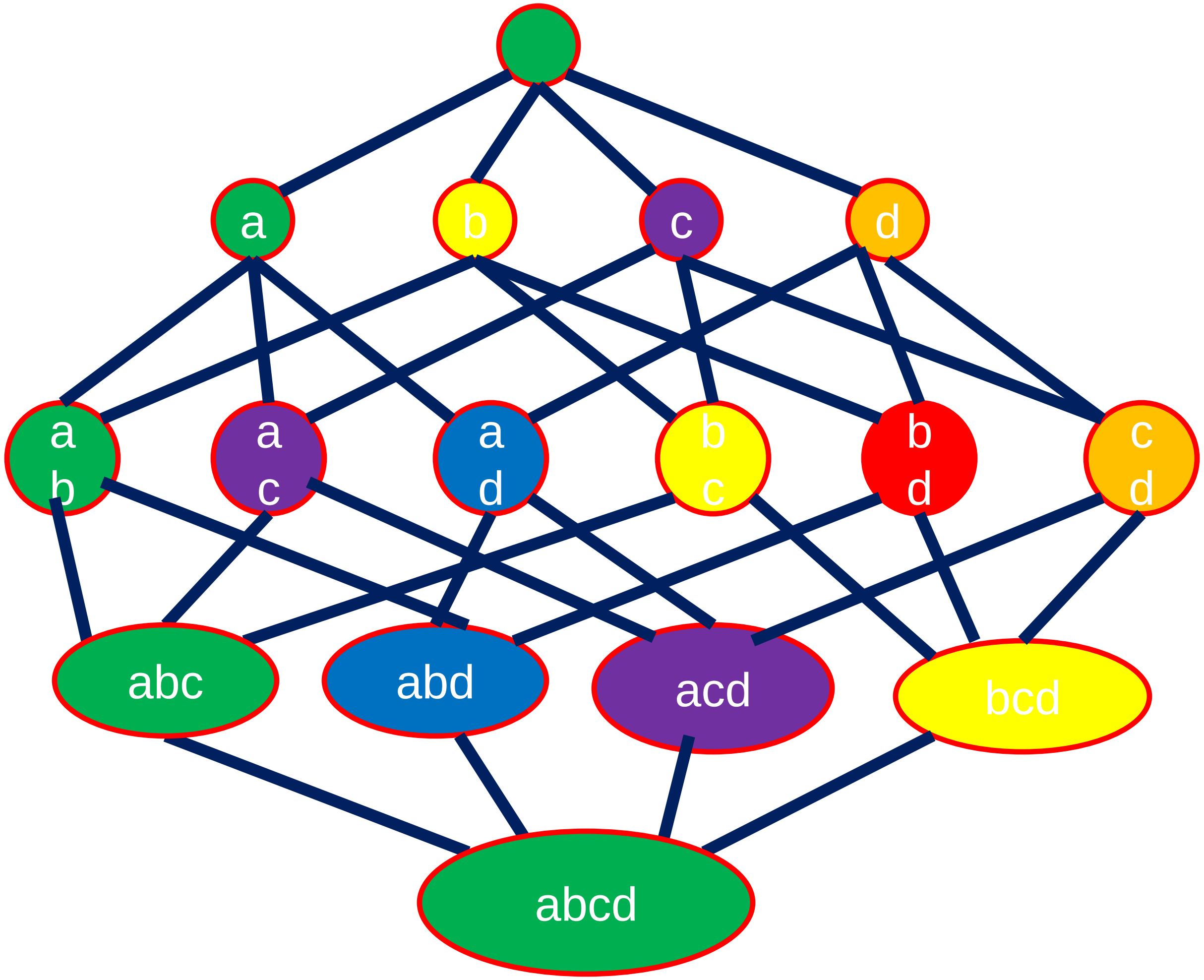
Chain and antichain

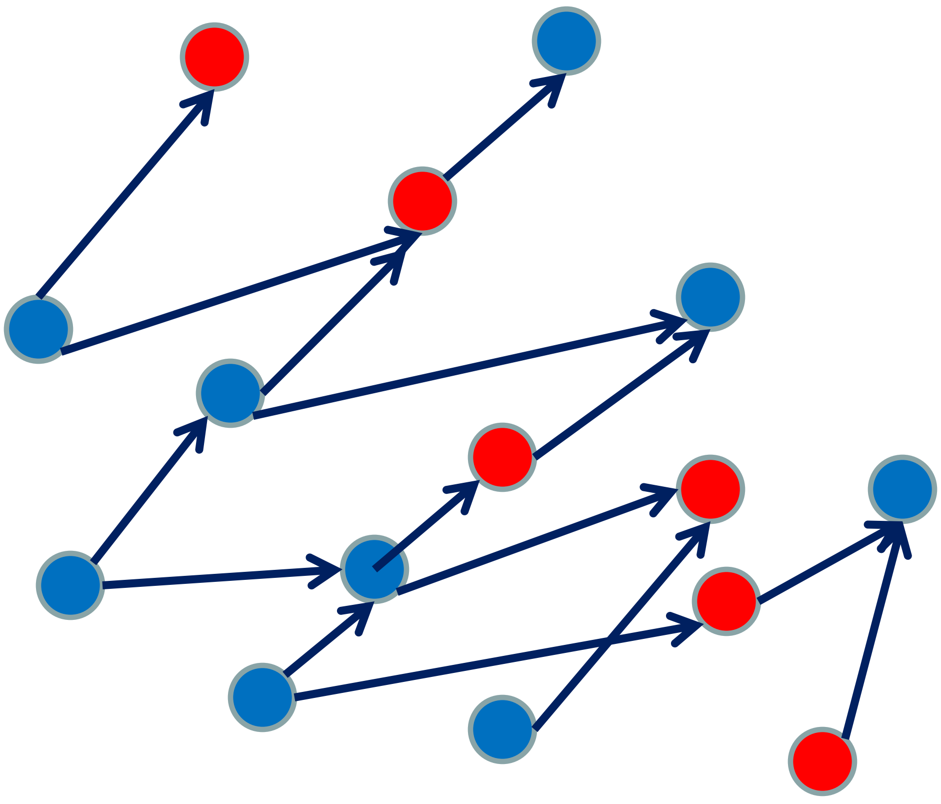
- Given a poset **A**, a chain of **A** is a sequence of elements

  a(1) < a(2) <….<a(k)
- An antichain is a set of uncomparable elements of **A**.
- μ(**A**): size of maximum antichain
- τ(**A**): size of minimum chain partition of **A**
- Exercise: show that μ(**A**)≦τ(**A**)



a
b
c
d
a b
a c
a d
b c
b d
c d
abc
abd
acd
bcd
abcd


a
b
c
d
a
b
a
c
a
d
b
c
b
d
c
d
abc
abd
acd
bcd
abcd



Another proof of Sperner's theorem

- If **A** is the powerr set P(S) of S, a Sperner system is an antichain

もしAがSの部分集合全体の集合なら、スペルナ系はアンチチェインになる。

- Prove Sperner's theorem from μ（**A**)≦τ（**A**)
  - Find a chain partition of size $_{n}C_{[n/2]}$

Dilworth's theorem

- Dilworth's theorem: $\mu(\mathbf{A})=\tau(\mathbf{A})$

A beautiful "duality" of chain and antichain

Dilworth's theorem

- Dilworth's theorem: $\mu(\mathbf{A})=\tau(\mathbf{A})$
- Applications:

1. Covering-matching duality
2. Hall's marriage theorem

   A Quiz on interconnecting network

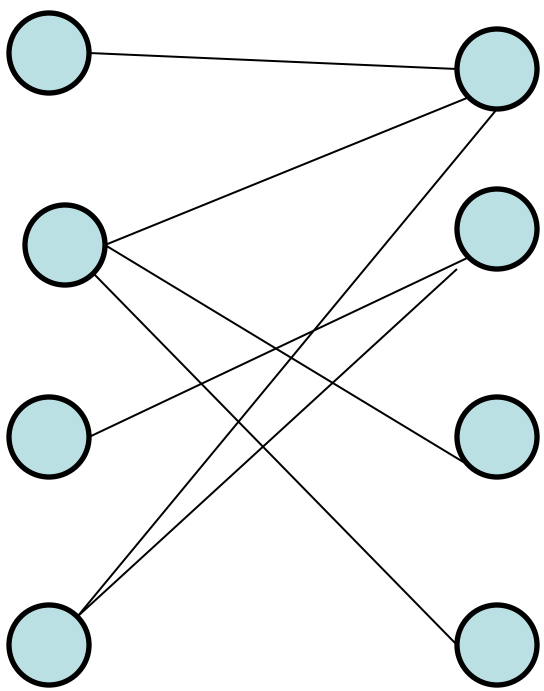
3. Refinment of Erdos-Szekeles theorem (next week??)
4. Young tableaux and its theory (without proofs)

Covering-matching duality

- Consider a bipartite graph G=(V,W,E)
- Matching : set of edges sharing no vertex.
  - If V is set of men and W is set of women, a set of married couples is a matching
- Covering: set of vertices X such that every edge has an endpoint in X
- Independent set: set of vertices Y such that none of them are adjacent
- Exercise: An independent set is the complement of a covering

Matching-covering duality

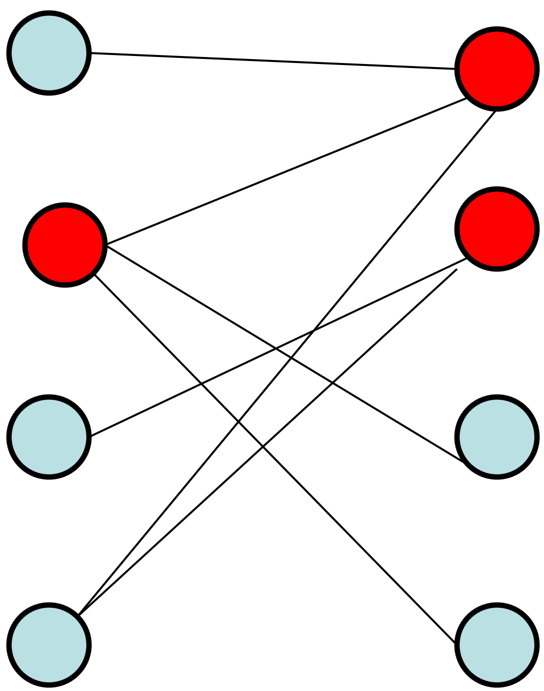
Koenig's Theorem: The maximum size of matching is the minimum size of covering



Does it has a matching of size 4?

Matching-covering duality

Koenig's Theorem: The maximum size of matching is the minimum size of covering

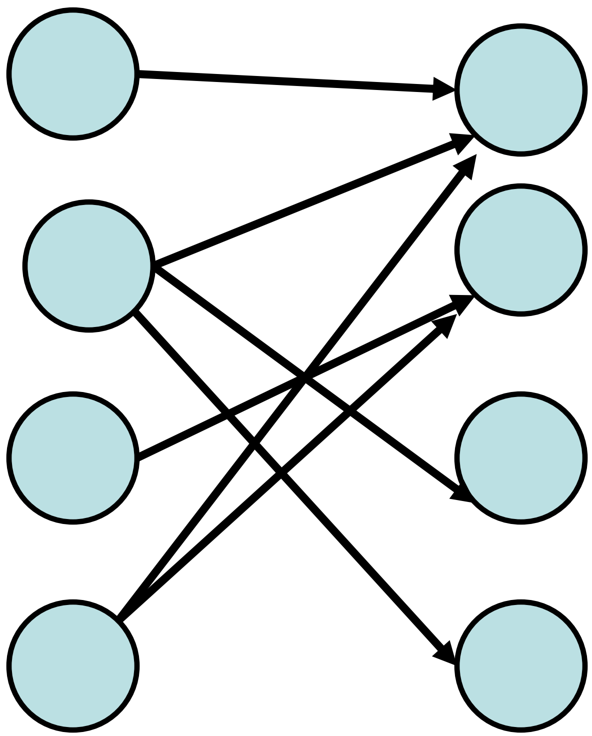


Does it has a matching of size 4?

No, because it has a covering of size 3.

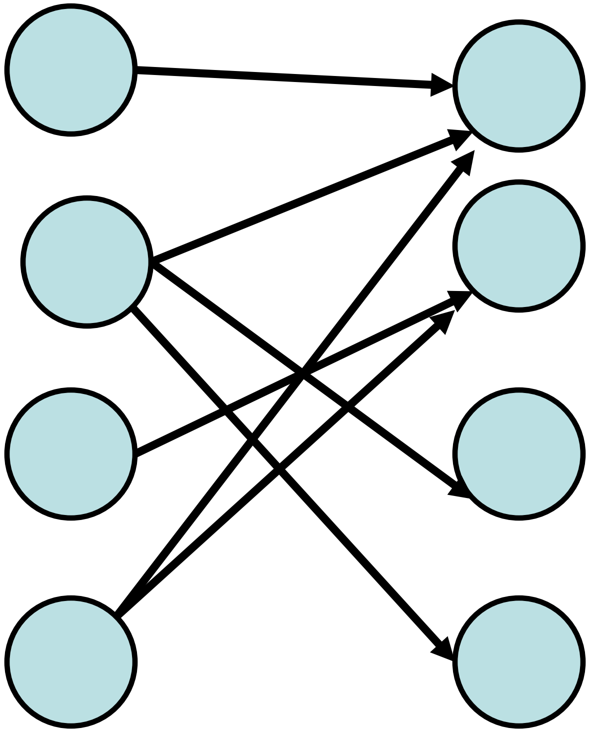
From Dilworth to Koenig

Consider the poset such that v<w if there is an edge (v,w) (from V to W).



From Dilworth to Koenig

Consider the poset such that v<w if there is an edge (v,w) (from V to W).



The minimum chain partition has cardinarity |V|+|W|-|M|, where M is a maximum matching

An antichain is an independent set

The maximum independent has size |V|+|W|-|C|, where C is the minimum cover

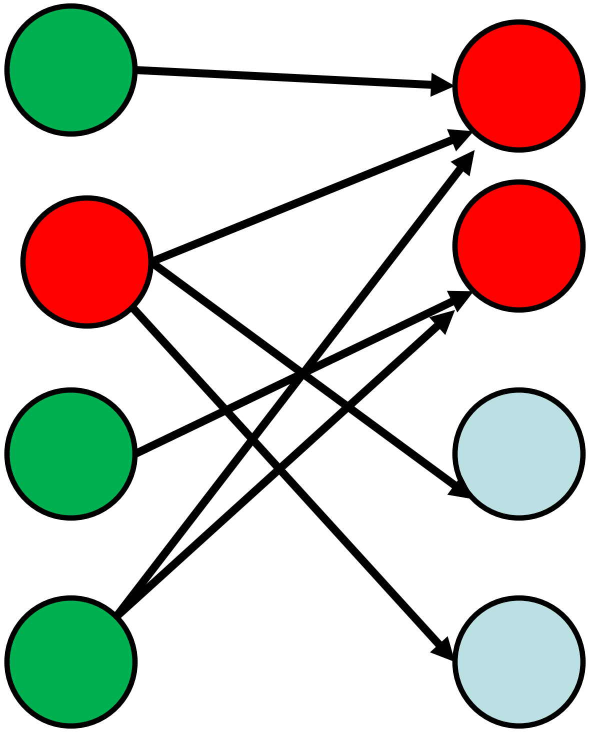
Thus, |M| =|C| from Dilworth's Th.

Hall's marriage theorem

- For G=(V, W,E) such that |V|≦|W|, it has a complete matching (i.e.matching of size |V|) if and only if for any subset A of V, |N(A)| ≧|A| holds, where N(A) is the neighbor of A in G.
- Proof. (only if) is easy. (if) is from Koenig.

Suppose that the max matching has size less than |V|. Then, there is a covering (V_0,W_0) of size at most |V|-1. Consider A = V – V_0 to contradict |N(A)| ≧|A|

Suppose that the max matching has size less than |V|. Then, there is a covering (V_0, W_0) of size at most |V|-1. Consider $A = V – V_0$ to contradict $|N(A)| \geqq |A|$



Nodes in A can only connect to vertices in W_0, since otherwise there is an uncovered edge by the covering.

Thus,
$|N(A)| \leqq |W_0| \leqq |V|-1-|V_0| < |A|$

Refined marriage theorem

- Let $\delta(A) = |A| - |N(A)|$ and its maximum among all subsets A of V is δ. Then, the size of maximum matching is $|V| - \delta$
- Proof is an exercise

Quiz

- There are 30 input lines connecting to 24 output lines. 10 inputs connects to only one output line, 10 inputs to two lines, and 10 inputs to four lines.

  Each output line is connected to at most three input lines.

  Prove that, we can establish 20 parallel connection irrelevant to the way of connection.

Hint: Show that $\delta(A) \leqq 10$ for any A.

Dilworth's theorem: let us prove

- Dilworth's theorem: μ（**A**)=τ（**A**)
- Definition: x ∈**A** is effective if there is an antichain of length μ(**A**) containing x.
- Lemma 1. Suppose that we have a chain partition M(1),M(2)..,M(k) of **A** for k=μ（**A**）, and let x(i) be the maximal effective element in M(i). Then, x(1),x(2),…,x(k) is an antichain.

Proof of Dilworth's theorem

- μ(**A**)=τ(**A**)
  - Show μ(**A**)$\geqq$$\tau$(**A**) by induction

1 . If |**A**| = 1, then easy

2. Remove a maximal element a from **A** to have **A**' =**A** - {a} : μ(**A**') = τ(**A**') = k by induction

3. Consider a maximum chain partition of **A**', and find x(1), x(2),..,x(k) of Lemma 1.

4. If a is not comparable to any x(i), fine (why?)

5. If a > x(1), -→ show at the blackboard.