Generalization of Superposition of Rectangles based on Direction Relations

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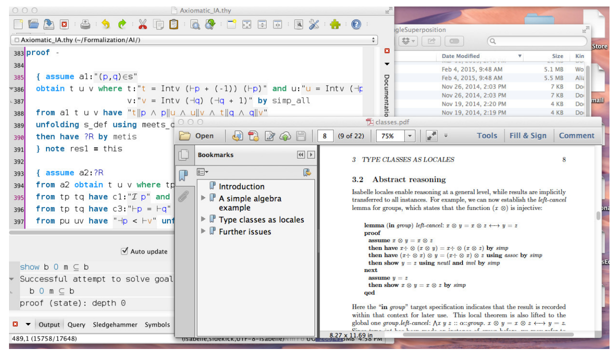
Abstract

We present a method of superposing rectangles. The superposition is under the condition that some of the regions should be visible. We first define a qualitative spatial representation of the rectangles. In particular, direction relations are used to express the positions of the must-be-visible regions. The representation is extendable to accommodate higher degree of granularity, and therefore to cover any arrangement of regions. Properties of success and effectiveness are defined to evaluate the superposition.

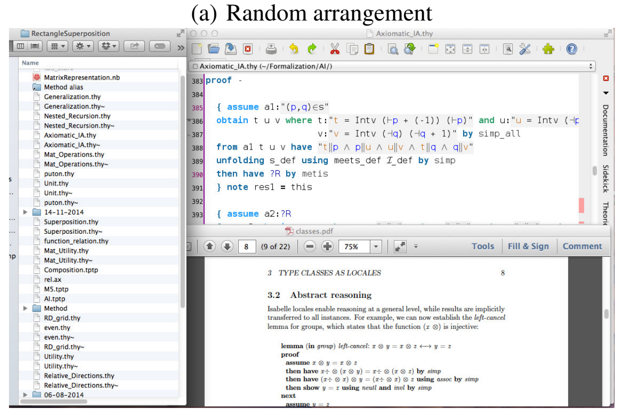
1 Introduction

Qualitative spatial representation emerged as an area of knowledge representation. The foundation in qualitative spatial representation is to treat objects of the space qualitatively, i.e. what matters is how objects are related. Positions of objects in the space is one of the relevant problems that is addressed by the field of qualitative spatial representation. The direction relations describe where an object is positioned w.r.t. a reference. We distinguish two categories of direction relations. The direction relations are established either w.r.t. a relative reference, in that case we use *relative direction relations* (Frank 1991), or w.r.t. an absolute reference, in that case we use *cardinal direction relations* (Clementini, Felice, and Hernándes 1997). The latter type of relations is used in the context of geographical space. They are the 9 classic relations *north*, *south*, *east*, *west*, *north_east*, *north_west*, *south_east*, *south_west* and *origin*. The relative direction relations are used in a local context and they are the 9 relations *up*, *bottom*, *right*, *left*, *up_right*, *up_left*, *bottom_right*, *bottom_left* and *same*.

In this paper, we use relative direction relations to not only represent positions of objects but also to treat their superpositions. We illustrate with a practically oriented problem, namely superposing rectangular structures. A tangible example is the arrangement of rectangular GUIs. When interacting with software, we often work with several GUIs at the same time. They are arranged randomly (e.g. Fig. 1(a)) or using tiling window manager without overlapping. However, the way the GUIs are arranged should depend on the content. We therefore often manually resize, drag and superpose the interfaces for better visibility (e.g. Fig. 1(b)). Our goal is to find solutions of superposing GUI rectangles while keeping important content visible.



(a) Random arrangement



(b) Manual arrangement where relevant content is visible

Figure 1: Arrangements of GUI windows with superposition

The rest of this paper is organized as follows. In Sect. 2, we summarize the original superposition method and its limitations. We describe the qualitative representation in Sect. 3. Then, in Sect. 4, we discuss the expressiveness, and we present the generalization of superposition and its properties in Sect. 5. In Sect. 6, we conclude with remarks on future directions of research.

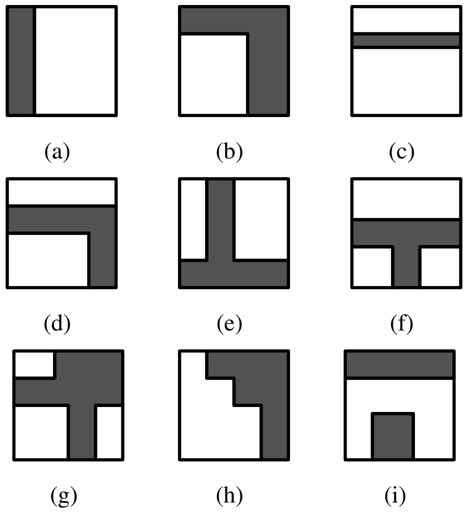


Figure 2: Various arrangements of black and white regions in rectangular units with a connected white region (Fig. 2(a), 2(b), 2(h) and 2(i)), and a disconnected white region (Fig. 2(c) - 2(g))

2 Superposition Method

The work presented in (Konishi and Takahashi 2012; Ghourabi and Takahashi 2015) covers qualitative arrangement of rectangles with partial superposition. A relevant content of rectangular GUI window is modelled as a white region that should be always visible. A superfluous content is a black region that can be hidden (c.f. Fig. 2 for examples of arrangements of white and black regions). The qualitative representation indicates the positioning of the white regions w.r.t. to the black regions. The defined superposition method proceeds by choosing and superposing parts of the black region. For instance, the superposition of the rectangle in Fig. 3(b) onto the one in Fig. 3(a) gives rise to the shape in Fig. 3(c). The black region around the bottom left corner in Fig. 3(b) is put on the black region around the upper right corner in Fig. 3(a). Before superposition, the rectangles can be resized so that their superposed regions fit. Since we treat this problem from qualitative spatial representation point of view, changing the size does not affect the qualitative representation. The direction relations between the rectangle's regions remain the same regardless of their size. At this point of our research, we ignore the effect that resizing has on the visibility of the content.

This qualitative representation of rectangles is formalized in the proof assistant Isabelle/HOL (Nipkow, Paulson, and Wenzel 2002). The direction relations between the regions are encoded in a matrix data-structure on which we perform operations of rotation and superposition. The superposition of rectangles is verified using proving capabilities of Isabelle/HOL (Ghourabi and Takahashi 2015).[1]

However, the above described method relies on few assumptions on the arrangement of white and black regions in each rectangle, and consequently the superposition is limited to rectangles whose structures can be represented using the 9 direction relations (29 rectangle structures in total). Examples of such rectangles are depicted in Fig. 2(a) - 2(f). The representation does not express more complex situations such as the ones depicted in Fig. 2(g) - 2(i). Such structures require extension of the set of direction relations to obtain precise information on the positions of regions. Furthermore, the superposition, when successful, generates only one solution. Besides the solution shown in Fig. 3(c), we can find 3 more depicted in Fig. 3(d) - 3(f).

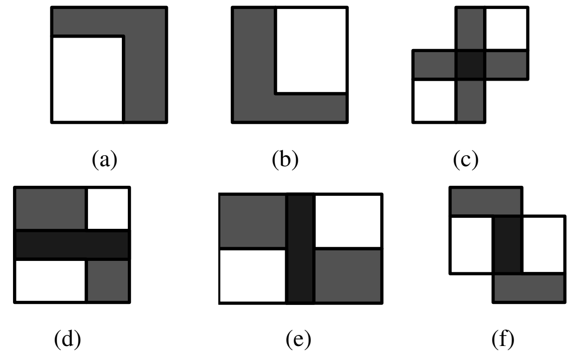


Figure 3: Different solutions of superposition

In an attempt to overcome the above issues, in this paper, we generalize the superposition method in (Konishi and Takahashi 2012) by relaxing all the assumptions, and hence allowing any GUI window structure. Nine direction relations are not suitable to describe precise positional positions inside any window. We therefore propose a qualitative representation with an extension of granularity level. The representation should be expressive yet compact, in the sense that it singles out only what constitutes an important positional information, and extensible, in the sense that it can accommodate precise information on spatial positions.

3 Tiling Approach

The qualitative representation in question provides qualitative information on the position of objects. It is determined by establishing the direction relations between spatial objects. In particular, the tiling approach in (Goyal and Egenhofer 1997; Chen et al. 2010; Li and Liu 2015) takes into consideration the shape of the objects and divides the plane into rectangular tiles, called also regions for direction relations. The *minimal bounding rectangle* (MBR) encloses the object, and its edges are extended to divide the plane into one bounded tile and 8 unbounded tiles. For instance, Fig. 4 shows the tiling of the plane resulting from extending the MBR of object A. The intersection of object C and the tiles to the left and bottom left are not empty, hence $dir(\text{A}, \text{C}) = \{left, bottom\_left\}$. Similarly, we deduce $dir(\text{A}, \text{D}) = \{same\}$. Issues may arise when two different objects are treated at the same position. In particular, when we deal with superposition of objects, the positions of A and D have to be distinguished. We present a variation of tiling approach

[1] For a closer look at the proofs of superposition method, the Isabelle/HOL theory files are available at `http://ist.ksc.kwansei.ac.jp/~ktaka/SuperpositionTheory/`.

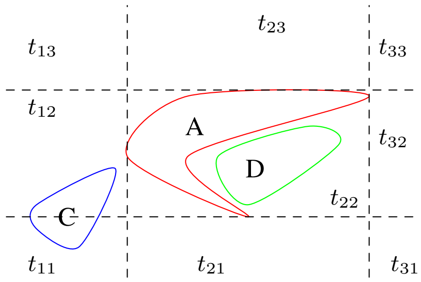


Figure 4: The MBR of object A and the tiling of the plane

that expresses direction relations between objects with better precision.

Subregions of a Unit

The spatial object of interest is a rectangular GUI window, called *unit*. The rectangular unit is composed of polygonal areas (convex or concave) that can be texts, pictures, videos, menu bar, etc. The polygons that have important content are modelled as W region, and should be always visible. What remains from the unit can be hidden, and it is modelled as B region. Examples of arrangements of W and B regions in a rectangular unit are depicted in Fig. 2. It is difficult, from observation alone, to find a solution for superposing the units in Fig. 2(d) onto Fig. 2(g) that keeps all the W regions visible.

We first define the qualitative representation of the units. The choice of tiling approach is natural. The W and B regions of a unit are polygons with horizontal and vertical edges. When we extend those edges, we divide the plane into bound and unbound tiles, and furthermore we split each region into smaller rectangles that we call subregions. A tile is a subregion of a unit as shown in Fig. 5.

Hereafter, we denote by ∇(W) the set of the subregions, i.e. the rectangular tiles, that make the W region. Similarly, ∇(B) is the set of the rectangles that make the B region. For the unit in Fig. 5, we have $\nabla(\mathrm{W}) = \{t_{11}, t_{13}, t_{23}\}$ and ∇(B) $= \{t_{21}, t_{12}, t_{22}\}$, where t_{ij} stands for the tile whose bottom left vertex is point (x_i, y_j). It is straightforward to see that the set ∇(W) and ∇(B) are disjoint and together they form the whole rectangular unit.

Representation

In order to compute the direction relation between two subregions t_{ij} and t_{pq}, function *dir* compares their bottom left vertices, i.e. (x_i, y_j) and (x_p, y_q).

$$dir(t_{ij}, t_{pq}) = \begin{cases} \{\mathit{same}\} & \text{if } x_i = x_p \text{ and } y_j = y_q \\ \{\mathit{up}\} & \text{if } x_i = x_p \text{ and } y_j < y_q \\ \{\mathit{bottom}\} & \text{if } x_i = x_p \text{ and } y_j > y_q \\ \{\mathit{right}\} & \text{if } x_i < x_p \text{ and } y_j = y_q \\ \{\mathit{up\_right}\} & \text{if } x_i < x_p \text{ and } y_j < y_q \\ \{\mathit{bottom\_right}\} & \text{if } x_i < x_p \text{ and } y_j > y_q \\ \{\mathit{left}\} & \text{if } x_i > x_p \text{ and } y_j = y_q \\ \{\mathit{up\_left}\} & \text{if } x_i > x_p \text{ and } y_j < y_q \\ \{\mathit{bottom\_left}\} & \text{if } x_i > x_p \text{ and } y_j > y_q \end{cases}$$

The position of W region w.r.t. a subregion $t_{ij} \in \nabla(\mathrm{B}) \cup \nabla(\mathrm{W})$ is therefore given by the set

$$dir(t_{ij}, \mathrm{W}) = \bigcup_{t_{pq} \in \nabla(\mathrm{W})} dir(t_{ij}, t_{pq}).$$

In Fig. 5, *dir*(t_{21}, t_{13}) = *dir*(t_{21}, t_{12}) = {*up_left*}. The locations of subregions t_{13} and t_{12} w.r.t. subregion t_{21} can be distinguished. Namely, t_{13} is located upper left to t_{22} which is upper up to t_{21}, we write *up_left* $\circ$ *up*. Here, the non-evaluated composition of direction relations in tiling approach describes the exact path to non connected tiles. The non-evaluated compositions of direction relations are relations in higher granularity level that we introduce in the next section.

Extension of the Granularity Level

Positions of objects can be represented in various granularity levels (Clementini, Felice, and Hernándes 1997; Moratz, Dylla, and Frommberger 2005). The nth granularity level defines n^2 direction relations that are used to express the positions of objects, where n is an odd natural number. In the case of the tiling approach, the extension of MBR partitions the plane into 9 regions that correspond to the 9 direction relations. Hence, we have 9 = 3^2 direction relations, which correspond to the 3rd granularity level.

Function dir_n computes the direction relations for the nth granularity level. In particular, dir_3 is a special case of function *dir*, and defined as follows.

$$dir_3(t_{ij}, t_{pq}) = \begin{cases} \{\mathit{same}\} & \text{if } x_i = x_p \text{ and } y_j = y_q \\ \{\mathit{up}\} & \text{if } x_i = x_p \text{ and } y_{j+1} = y_q \\ \{\mathit{bottom}\} & \text{if } x_i = x_p \text{ and } y_{j-1} = y_q \\ \{\mathit{right}\} & \text{if } x_{i+1} = x_p \text{ and } y_j = y_q \\ \{\mathit{up\_right}\} & \text{if } x_{i+1} = x_p \text{ and } y_{j+1} = y_q \\ \{\mathit{bottom\_right}\} & \text{if } x_{i-1} = x_p \text{ and } y_{j+1} = y_q \\ \{\mathit{left}\} & \text{if } x_{i-1} = x_p \text{ and } y_j = y_q \\ \{\mathit{up\_left}\} & \text{if } x_{i-1} = x_p \text{ and } y_{j+1} = y_q \\ \{\mathit{bottom\_left}\} & \text{if } x_{i-1} = x_p \text{ and } y_{j-1} = y_q \end{cases}$$

Let $n = 2s+1$ where $s > 1$, function dir_n is then defined inductively as follows.

$$dir_{2s+1}(t_{ij}, t_{pq}) = \\ \begin{cases} dir_3(t_{ij}, t_{pq}) & \text{if defined} \\ dir_3(t_{kl}, t_{pq}) \circ dir_{2(s-1)+1}(t_{ij}, t_{kl}) & \text{otherwise} \end{cases}$$

The direction relations of the fifth granularity level are the 25 relations depicted in Fig. 6.

4 Expressivenes

In this section, we discuss how to decide the granularity level in which the positions are determined. Let H_U and V_U be the number of the horizontal and vertical lines obtained by extending the edges of W and B regions of the unit U. The granularity level that allows describing precise relations between all the subregions is the minimum granularity level, denoted by n_{min}, such that $max(H_U, V_U) < n_{min}$. For instance, in Fig. 5, we have $max(4, 3) < 5$, hence the 5th granularity level ensures complete description of all the positions.

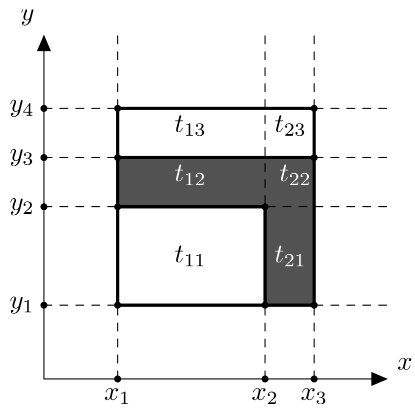


Figure 5: The tiling of a unit into rectangular subregions

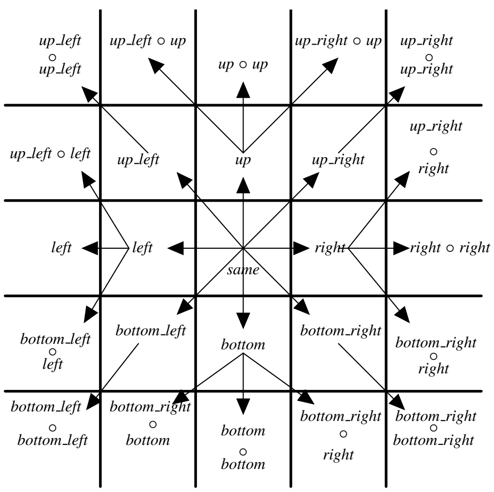


Figure 6: Direction relations of the 5th granularity level

However, it is not reasonable to represent positions at precise level for all the pairwise combinations of subregions. Large set of relations may emerge as consequence. What we need to represent with precision is what we think it is an important information. In the case of superposition of units, the positions of W subregions are important, since they should be kept visible. To that end, we first introduce the cores of a unit.

Definition 1 (Core)**.** *The* cores *of a unit w.r.t. the granularity level n are the B-subregions $t_{ij} \in \nabla$(B) with maximum* $|dir_n(t_{ij}, W)|$.

A core of a unit is a subregion of B region that captures maximum information on the positions of the subregions of W region. If the maximum information is, furthermore, all the precise positions of W region, then we say that the representation is W-expressive.

Definition 2 (W-expressiveness)**.** *Let $c_{U,n} \in \nabla(B)$ be a core of unit U w.r.t. the nth granularity level. If $|dir_n(c_{U,n}, W)| = |\nabla(W)|$, then dir_n is W-expressive.*

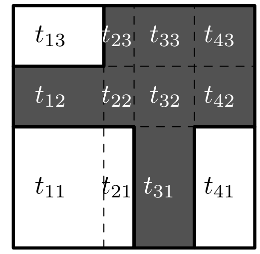


Figure 7: A unit that requires representation in the 5th granularity level

For example, for the unit in Fig. 5, we have $dir_3(t_{22}, \mathrm{W}) = \{up, up\_left, bottom\_left\}$, $dir_3(t_{12}, \mathrm{W}) = \{up, up\_right, bottom\}$ and $dir_3(t_{21}, \mathrm{W}) = \{left\}$. Hence, t_{22} and t_{12} are the cores of the unit. Although dir_3 does not provide precise positions, we still can find a core that provides the position of all subregions of W region. In particular, $|dir_3(t_{22}, \mathrm{W})| = |dir_3(t_{12}, \mathrm{W})| = |\nabla(\mathrm{W})| = 3$. Then, the qualitative representation is W-expressive for the 3rd granularity level.

Now, we consider the unit in Fig. 7. The cores are subregions t_{12} and t_{22}. For t_{41} a subregion of W region, both of $dir_3(t_{12}, t_{41})$ and $dir_3(t_{22}, t_{41})$ are undefined. The qualitative representation using dir_3 is not W-expressive, and here the extension to a higher level is necessary.

5 Generalization of Superposition

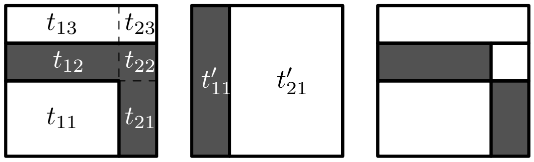
The superposition proceeds by putting one unit above the other while keeping W region visible. In this paper, we focus on the generalization of the method presented in (Konishi and Takahashi 2012) that operates by putting a core of unit U_2 on a core of a unit U_1. Let n be the granularity level in which dir_n is W-expressive. Let $c_{1,n}$ and $c_{2,n}$ be two cores of units U_1 and U_2. A superposition is defined by a pair of cores ($c_{1,n}$, $c_{2,n}$), which means that the superposition of U_2 onto U_1 is obtained by placing the core $c_{2,n}$ on the core $c_{1,n}$.

Figure 8 illustrates the superposition of unit U_2 onto unit U_1. The core $c_{2,3} = t'_{11}$ is placed on the core $c_{1,3} = t_{12}$ as shown in Fig. 8(c).[2] Since the cores are not necessary of the same size, units U_1 and U_2 are scaled so that their cores fits. This operation does not affect the qualitative representation of the positional information. There are situations where resizing a unit leads to a small W region to the point where it becomes unusable. Although GUI placement is a motivation of this work but the objective is rather the qualitative method of partial superposition of rectangles and its properties.

Result of Superposition

After superposing a unit U_2 onto a unit U_1, all the subregions of U_2 are in the foreground and some of the subregions of U_1 are hidden. We determine the sets S_W and S_B of the

[2](t_{12}, t'_{11}) is not the only possible superposition pair. We can select the core t_{22} of U_1 and perform the superposition defined by the pair (t_{22}, t'_{11}).



(a) U_1 (b) U_2 (c) U_2 on U_1

Figure 8: Superposition in the 3rd granularity level

subregions that are visible after superposition.

$$S_W = [\nabla(\mathrm{W})]_{U_2} \cup [\nabla(\mathrm{W})]_{U_1}$$
$$S_B = [\nabla(\mathrm{B})]_{U_2} \cup ([\nabla(\mathrm{B})]_{U_1} \backslash \{s \mid s \in [\nabla(\mathrm{B})]_{U_1} \wedge dir_n(c_{1,n}, s) \in dir_n(c_{2,n}, \mathrm{W}) \cup dir_n(c_{2,n}, \mathrm{B})\}),$$

where $[\nabla(\mathrm{B})]_X$ and $[\nabla(\mathrm{W})]_X$ denote the subregions of unit X.

The relations established by function dir_n are independent from the change of sizes of U_1 and U_2. The subregions of U_3, the result of superposition, are sets $[\nabla(\mathrm{B})]_{U_3}$ and $[\nabla(\mathrm{W})]_{U_3}$ computed from scaling the subregions in the sets S_B and S_W.

Properties of Superposition

Superposing a unit U_2 onto a unit U_1 is not always possible. We have to check that the hidden subregions are not part of the W region. In other words, the positions of W region w.r.t. core $c_{1,n}$ (i.e. $dir_n(c_{1,n}, \mathrm{W})$) do not overlap with the positions of W and B regions w.r.t. core $c_{2,n}$ (i.e. $dir_n(c_{2,n}, \mathrm{W}) \cup dir_n(c_{2,n}, \mathrm{B})$).

Definition 3 (Success)**.** *A superposition defined by a pair of cores ($c_{1,n}$, $c_{2,n}$) in the nth granularity level is* successful*, iff $dir_n(c_{1,n}, W) \cap (dir_n(c_{2,n}, W) \cup dir_n(c_{2,n}, B)) = \emptyset$.*

Note that the above condition is w.r.t. the granularity level in which the representations are expressed. In case the success condition does not hold for any pair of cores in the nth granularity level, we may find a solution in a higher granularity level. We need to examine the condition of successfulness in higher granularity level until either a solution is found or we reach n_{min} (cf. Example 2).

Different pairs of cores may lead to successful superpositions. The choice is made on the pair of cores that hide the most of B region. We introduce two orders $\approx$ and $\ll$ to compare superpositions pairs. Namely, we have

- ($c_{1,n}$, $c_{2,n}$) $\approx$ ($c'_{1,n}$, $c'_{2,n}$), if $|S_B| = |S'_B|$, and
- ($c_{1,n}$, $c_{2,n}$) $\ll$ ($c'_{1,n}$, $c'_{2,n}$), if $|S_B| < |S'_B|$.

Definition 4 (Effectiveness)**.** *Let ($c_{1,n}$, $c_{2,n}$) and ($c'_{1,n}$, $c'_{2,n}$) be two pairs of successful superpositions of unit U_2 onto U_1. The superposition defined by the pair ($c_{1,n}$, $c_{2,n}$) is* more effective *than the one defined by the pair ($c'_{1,n}$, $c'_{2,n}$) if it hides more B subregions, i.e. if ($c_{1,n}$, $c_{2,n}$) $\ll$ ($c'_{1,n}$, $c'_{2,n}$).*

Based on the position information in each unit, we can decide the property of success and effectiveness before performing the superposition.

Example 1.

We consider U_1 and U_2 the units in Fig. 9(a) and Fig. 9(b), respectively. As we explained in Sect. 3, for unit U_1, dir_3 is not W-expressive. We therefore extend to the 5th granularity level and deduce that t_{31}, t_{22}, t_{32}, t_{23} and t_{33} are cores.

The positions of W and B regions w.r.t. the subregions in $[\nabla(\mathrm{B})]_{U_2}$ in the 5th granularity level.

$dir_5(t'_{11}, \mathrm{B}) = \{right, up\_right, right \circ right, same\}$
$dir_5(t'_{11}, \mathrm{W}) = \{up, up\_right \circ right\}$
$dir_5(t'_{21}, \mathrm{B}) = \{up, left, right, same\}$
$dir_5(t'_{21}, \mathrm{W}) = \{up\_left, up\_right\}$
$dir_5(t'_{31}, \mathrm{B}) = \{left, up\_left, left \circ left, same\}$
$dir_5(t'_{31}, \mathrm{W}) = \{up, up\_left \circ left\}$
$dir_5(t'_{22}, \mathrm{B}) = \{bottom, bottom\_right, bottom\_left, same\}$
$dir_5(t'_{22}, \mathrm{W}) = \{left, right\}$

From the above equalities, we deduce the cores t'_{11}, t'_{21}, t'_{31}, t'_{22}. The successfulness condition holds for the superposition pairs (t_{22}, t'_{11}), (t_{32}, t'_{11}), (t_{32}, t'_{21}), (t_{42}, t'_{11}), (t_{42}, t'_{21}), (t_{42}, t'_{31}), (t_{23}, t'_{11}), (t_{33}, t'_{11}), (t_{33}, t'_{21}), (t_{43}, t'_{11}), (t_{43}, t'_{31}) and (t_{43}, t'_{22}).

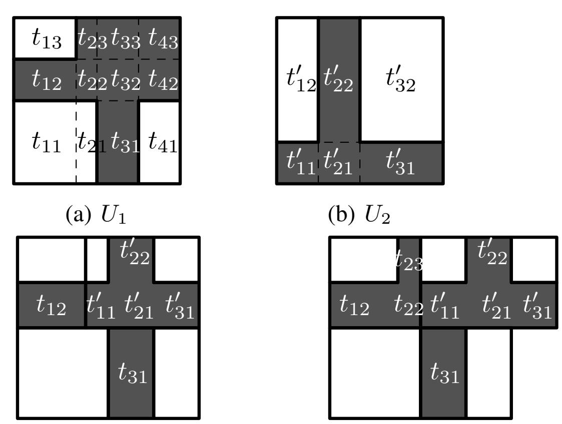
Let's consider the pairs (t_{32}, t'_{21}) and (t_{32}, t'_{11}) depicted in Figs. 9(c) and 9(d), respectively. The sets S_{B1} and S_{B2} of the superposition pairs are $[\nabla(B)]_{U_2} \cup \{t_{31}, t_{12}\}$ and $[\nabla(B)]_{U_2} \cup \{t_{31}, t_{12}, t_{22}, t_{23}\}$, respectively. We have $|S_{B1}| = 6 < |S_{B2}| = 8$. Hence, ($t_{32}$, t'_{21}) $\ll$ (t_{32}, t'_{11}), and the superposition defined by the pair (t_{32}, t'_{21}) is more effective than the superposition defined by the pair (t_{32}, t'_{11}).

Example 2.

In this example, we superpose U_2 on U_1 in Fig. 10. dir_3 is W-expressive, and t_{12} and t'_{11} are cores of U_1 and U_2, respectively.

$dir_3(t_{12}, \mathrm{W}) = \{up, up\_right, right, bottom, bottom\_right\}$
$dir_3(t'_{11}, \mathrm{W}) = \{right\}$
$dir_3(t'_{11}, \mathrm{B}) = \{same\}$

The condition of success does not hold for the superposition pair (t_{12}, t'_{11}) at granularity level 3. Namely, we have $dir_3(\mathrm{W}, t_{12}) \cap (dir_3(\mathrm{W}, t'_{11}) \cup dir_3(\mathrm{B}, t'_{11})) = \{right\}$. When extending to the 5th granularity level, all the B subregions of U_1 are cores. We consider the superposition pair (t_{31}, t'_{11}).



(a) U_1 (b) U_2

(c) Superposition (t_{32}, t'_{21}) (d) Superposition (t_{32}, t'_{11})

Figure 9: Superposition with extension of granularity

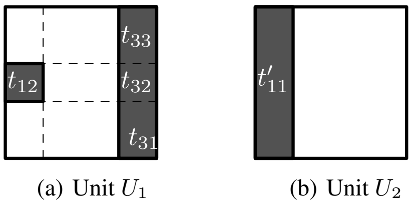


Figure 10: No solution for the superposition of unit U_2 onto U_1 at the 3rd granularity level

$dir_5(t_{31}, \mathrm{W}) =$
$\{left, up\_left, up\_left \circ up, up\_left \circ up\_left, left \circ left\}$
$dir_5(t'_{11}, \mathrm{W}) = \{right\}$
$dir_5(t'_{11}, \mathrm{B}) = \{same\}$

It is straightforward that $dir_5(t_{31}, \mathrm{W}) \cap (dir_5(t'_{11}, \mathrm{W}) \cup dir_5(t'_{11}, \mathrm{B})) = \emptyset$. Therefore the pair (t_{31}, t'_{11}) leads to successful superposition.

6 Conclusion

We proposed a generalization of the qualitative representation of rectangular unit to overcome limitations in the original superposition method (Ghourabi and Takahashi 2015). The qualitative representation is a variation of the tiling approach, and takes into account the polygonal shape of the regions of a unit. We defined the extension of granularity level that allows better expressiveness of the qualitative representation. As a result, the generalization covers the superposition of any unit structure. We evaluate the superposition of units by checking properties of success and effectiveness.

This paper presents a preparatory but necessary generalization of superposition towards its formalization in proof assistant. As further research, we plan to formalize the qualitative representation based on granularity levels of direction relations.

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