A Revised Method on Judging Abstract Shape of a Curve on a Two-Dimensional Plane *

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Abstract. This paper discusses the representation and reasoning of spatial data in a qualitative manner. Previously, we proposed a symbolic expression to represent the abstract shape of a curve on a two-dimensional plane. In this expression, the curve is generated by connecting multiple primitive segments, each characterized by an abstract direction and curvature. A method was proposed to determine whether the curve forms a spiral, which is unsuitable as a border for a natural object. However, this method has drawbacks. In this paper, we revise the method and prove its properties. In addition, we demonstrate that it is possible to draw a curve in a specified abstract shape using the proposed expression.

Keywords: qualitative spatial reasoning, knowledge expression, logic for reasoning under uncertainty

1 Introduction

Recognizing objects or regions in images and videos is often required in various research fields. Several images may be blurry or noisy, and because of the projection of three-dimensional objects onto a two-dimensional plane, some parts may be occluded. As a result, complete data is rarely obtained. Currently, the primary approach for identification in such incomplete data uses machine learning techniques. However, when employing machine learning, a large amount of data is necessary to achieve accurate results. Additionally, the results lack explanations; although research in explainable AI (XAI) is advancing, the generated explanations are still insufficient to fully convince users. On the other hand, alternative approaches exist that handle incomplete data as it is. One of these is a method based on Qualitative Spatial Reasoning (QSR).

QSR is a symbolic approach that represents and reasons about spatial entities without using numerical data [3, 10, 2, 12]. The spatial aspects required by the user are extracted and given in a symbolic expression, and reasoning is conducted on this expression. It is advantageous for roughly capturing spatial properties of objects or the relationships between objects, and for avoiding human errors caused by using precise data. For example, in the framework of egg-yolk, a region with a vague or indeterminate boundary is represented by two crisp regions,

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called the egg and the yolk. It is not determined whether the points between the egg and yolk are in such an ambigiuous region or not [6, 7]. QSR also fits human recognition, provides explanations for derived results, and reduces computational burdens since it does not use numerical data or require big data.

Takahashi has proposed an approach based on QSR to handle incomplete and ambiguous spatial data [13, 14]. Their target was continuous open curves projected onto a two-dimensional plane, which correspond to the borderlines of natural regions, such as geological strata. They divide the curve into multiple segments at inflection points and extremum points, and represent each segment using its abstract direction and curvature. They also determine missing parts of the curve by connecting such segments to reconstruct its abstract shape. They demonstrated the method for judging whether the curve, obtained in this manner, forms a spiral by checking the orientation of the segments that constitute it. However, the method has some drawbacks. In this paper, we revise the method and show that it is possible to draw a curve on a two-dimensional plane with an abstract shape derived from the expression.

This paper is organized as follows. In Section 2 and Section 3, we describe the formalization and embedding of a curve on a two-dimensional plane, respectively, that were proposed in our previous papers [14, 13]. In Section 4, we present the revised judgment method and prove its property. In Section 5, we compare our approach with related works. Finally, in Section 6, we show our conclusions and future works.

2 Fundamental Concepts

Let CURVES be a set of directed curved segments with a unique direction and curvature on a two-dimensional plane. For $\boldsymbol{X} \in \mathit{CURVES}$, we represent the qualitative shape of $\boldsymbol{X}$ focusing on its intrinsic direction and convexity, ignoring the precise size and the exact curvature.

Let $S_v = \{n, s\}$, $S_h = \{e, w\}$, $\mathit{Conv} = \{cx, cc\}$ and $\mathit{Dir} = S_v \cup S_h$. The symbols n, s, e and w indicate the north, south, east and west directions, respectively, and cx and cc indicate convex and concave, respectively. The direction exactly in the middle between north and south (east and west) is regarded as either n or s (e or w, resp.). Straight lines are not considered. For $\boldsymbol{X} \in \mathit{CURVES}$, $X = (V, H, C)$ is said to be *the qualitative representation of* $\boldsymbol{X}$ where $V \in S_v, H \in S_h$ and $C \in \mathit{Conv}$. V, H and C show the vertical direction, horizontal direction and the convexity of $\boldsymbol{X}$, respectively. For $\boldsymbol{X}, \boldsymbol{Y} \in \mathit{CURVES}$, let $X = (V, H, C)$ and $Y = (V', H', C')$ be qualitative representations of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. We define the relation $\sim$ on CURVES as follows: $\boldsymbol{X} \sim \boldsymbol{Y}$ iff $V = V'$, $H = H'$ and $C = C'$. Then $\sim$ is an equivalence relation on CURVES. As a result, CURVES is classified into eight equivalence classes which are jointly exhaustive and pairwise disjoint. We denote the set of these equivalence classes as $\mathcal{S}$, that is, $\mathcal{S} = \mathit{CURVES}/\sim$. Then, $\boldsymbol{X} \in \mathit{CURVES}$ is mapped to $X \in \mathcal{S}$.

For example, in Figure 1, the three segments in the upper frame are regarded as equivalent, whereas they are not equivalent to the lower two segments. The qualitative representation of a segment in the upper frame is (n, e, cx).

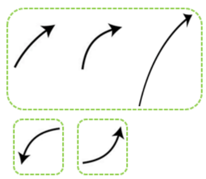


Fig. 1. Classes of curved segments.

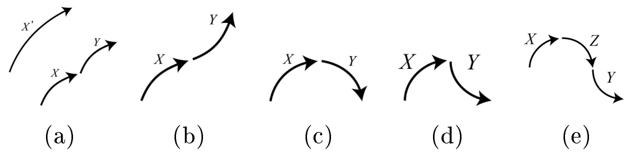


Fig. 2. Connection of segments.

In this paper, a smooth continuous curve without a self-intersection is called an *scurve*. We connect multiple segments in $\mathcal{S}$ to create an *scurve*.

For $X \in \mathcal{S}$, its initial and terminal points are indicated by $init(X)$ and $term(X)$, respectively. For $X, Y \in \mathcal{S}$, if an *scurve* is obtained by considering that $init(Y)$ and $term(X)$ are identical, then X and Y are said to be *directly connectable*, denoted by $dc(X, Y)$, and the outcome of the connection is represented as $X \cdot Y$. For $X, Y \in \mathcal{S}$, if $X = Y$, then they are directly connectable and the result is regarded as a single segment without a cusp, since the precise curvatures of X and Y are ignored (Figure 2(a)). When X and Y are directly connectable, and $X \neq Y$, the connection of X and Y creates an inflection point (Figure 2(b)), or an extremum point (Figure 2(c)) via direct connections. For $X_1, \ldots, X_n \in \mathcal{S}$ $(n \geq 2)$, if for each j such that $1 \leq j \leq n-1$, $dc(X_j, X_{j+1})$, then we obtain an *scurve* by directly connecting X_j and X_{j+1}, and the outcome of the connections is represented as $X_1 \cdot \ldots \cdot X_n$. As a result, *scurve* is a sequence of qualitative representations of curved segments. For example, $X = (n, e, cx)$ and $Y = (s, e, cc)$ are not directly connectable, since a cusp is created at their connection (Figure 2(d)); but if we insert $Z = (s, e, cx)$ between X and Y, then we get an *scurve* $X \cdot Z \cdot Y$ (Figure 2(e)).

3 Embedding of an *scurve* on a Two-Dimensional Plane

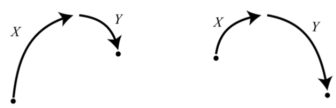
In the following, 'embedding of X' means an assignment of one $\boldsymbol{X} \in \mathit{CURVES}$ to $X \in \mathcal{S}$.

Definition 1 (embedding).

1. *Let* $\boldsymbol{X} \in \mathit{CURVES}$ *be a curved segment on a two-dimensional plane of which* $X \in \mathcal{S}$ *is its qualitative representation. Then* $\boldsymbol{X}$ *is said to be* an embedding of X. $init(\boldsymbol{X})$ *and* $term(\boldsymbol{X})$ *represent the locations of the initial point and the terminal point of* $\boldsymbol{X}$ *on a two-dimensional plane, respectively.*

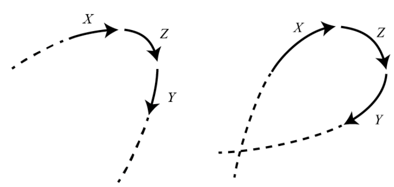
2. Let $X_1 \cdot \ldots \cdot X_n$ *be an* scurve $X_1 \cdot \ldots \cdot X_n$*, and* $\boldsymbol{X}_i$ *(*$\forall i; 1 \leq i \leq n$*) be* an embedding of X_i*. For all* i *such that* $1 \leq i \leq n-1$*, if* $term(\boldsymbol{X}_i)$ *and* $init(\boldsymbol{X}_{i+1})$ *are located in the same position, then* $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_n$ *is said to be* an embedding of an *scurve* $X_1 \cdot \ldots \cdot X_n$.

Note that there are infinite number of $\boldsymbol{X}$'s for $X \in \mathcal{S}$. For example, Figure 3 shows two kinds of embedding of $X \cdot Y$ where $X = (n, e, cx)$ and $Y = (s, e, cx)$. The relative directions of the locations of $term(\boldsymbol{Y})$ with respect to $init(\boldsymbol{X})$ are (n, e) and (s, e) in Figure 3(a) and Figure 3(b), respectively.



(a) (b)

Fig. 3. Different embeddings of $X \cdot Y$.



(a) (b)

Fig. 4. Open/closed embedding

For an embedding of $\boldsymbol{X}$, $dir(\boldsymbol{X})$ is represented as (V, H) where $V \in S_v$ and $H \in S_h$, which indicates the relative direction of $term(\boldsymbol{X})$ with respect to $init(\boldsymbol{X})$. For an embedding of $\boldsymbol{X} \cdot \boldsymbol{Y}$, $dir(\boldsymbol{X} \cdot \boldsymbol{Y})$ is represented as (V, H) which indicates the relative direction of $term(\boldsymbol{Y})$ with respect to $init(\boldsymbol{X})$. If $X \cdot Y$ creates an inflection point, then $dir(\boldsymbol{X} \cdot \boldsymbol{Y}) = dir(\boldsymbol{X}) = dir(\boldsymbol{Y})$; while if it creates an extremum point, it is nondeterministic.

If an embedding of an *scurve* forms a spiral, it is not desirable, when an *scurve* corresponds to a boundary of an actual object. However, there exists an *scurve* which cannot be drawn in a non-spiral form no matter how it is drawn. We show how to determine whether there exists an embedding that does not form a spiral on a two-dimension plane, for a given *scurve*. For this purpose, we introduce a concept of open/closed embedding.

Definition 2 (closed, open). *For* $X, Y \in \mathcal{S}$*, let* C *be an embedding of an* scurve *from* X *to* Y *on a two-dimensional plane, where* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are embeddings of* X *and* Y*, respectively. And let* C' *be an infinite-length curve that is obtained by extending* C *in both directions in a manner such that the curvatures of* $\boldsymbol{X}$ *at* $init(\boldsymbol{X})$ *and* $\boldsymbol{Y}$ *at* $term(\boldsymbol{Y})$ *are preserved. If* C' *has a self-intersection point, then the embedding is said to be* closed*; otherwise, it is* open*.*

Here, a closed embedding is considered to be a spiral form.

Definition 3 (admissible). *If there is an open embedding of an* scurve*, then the* scurve *is said to be* admissible*.*

For example, Figure 4(a) and Figure 4(b) show open embedding and closed embedding of *scurve* $X \cdot Z \cdot Y$, respectively, where $X = (n, e, cx)$, $Z = (s, e, cx)$, $Y = (s, w, cc)$. Therefore, $X \cdot Z \cdot Y$ is admissible. On the other hand, $X \cdot Z \cdot Y \cdot W$, where $W = (n, w, cc)$ is not admissible.

4 Judgment of Admissibility

4.1 Orientation of an *scurve*

We use an orientation of an *scurve* to judge its admissibility.

For $X \in \mathcal{S}$, its orientation is defined either as clockwise $(+)$ or anti-clockwise $(-)$. Moreover, the orientation of an *scurve* is defined as a sequence of orientations of the segments that configure it. The function *inv* is the assignment of the opposite orientation, that is, $inv(+) = -$ and $inv(-) = +$. We denote a set of orientations for *scurves* by Σ.

Definition 4 (orientation).

- *For* $X \in \mathcal{S}$,
  $orn(X) = '+'$ *iff* $X = (n, e, cx)$, (s, e, cx), (s, w, cc) *or* (n, w, cc); $orn(X) = '-'$ *iff* $X = (s, w, cx)$, (s, e, cc), (n, e, cc) *or* (n, w, cx).
- *For* $X_1, \ldots, X_n \in \mathcal{S}$,
  $orn(X_1 \cdot \ldots \cdot X_n) = orn(X_1) \ldots orn(X_n)$.

Definition 5 (rotation difference). *For an* scurve p*, the difference of the numbers of* $+$ *and* $-$ *that appear in* $orn(p)$ *is said to be* rotation difference of p*.*

4.2 Reduction

Generally it is known that if the rotation angle of a curve is greater than or equal to 2π, then it forms a spiral and may have a self-intersection point on a two-dimensional plane. For an *scurve* p, if its rotation difference is more than three, the rotation angle of p is greater than or equal to 2π; in this case, p is not admissible. Then, how can we determine the admissibility in the other cases? For example, is the *scurve* of which an orientation is $-++-++++--$ admissible? To address this problem, the reduction has been introduced [14]. However, the proposed definition has two drawbacks: a redundant condition is imposed on applying the rule, and confluence on the reduction process does not hold. Here, we revise the definition, and show its properties with their correctness.

There are two reduction rules: the subsequence $+-+$ (or $-+-$) is reduced to $+$ (or $-$, resp.), and the subsequence $++--$ (or $--++$) is reduced to $+-$ (or $-+$, resp.).

[reduction rule]

Let $\sigma_1, \sigma_2 \in \Sigma$ and $s_1, s_2, s_3, s_4 \in \{+, -\}$.

(r1) If $s_1 = s_3 = inv(s_2)$, then $\sigma_1 s_1 s_2 s_3 \sigma_2$ is reduced to $\sigma_1 s_1 \sigma_2$.

(r2) If $s_1 = s_2 = inv(s_3) = inv(s_4)$, then $\sigma_1 s_1 s_2 s_3 s_4 \sigma_2$ is reduced to $\sigma_1 s_1 s_4 \sigma_2$.

Definition 6 (reduced form). *For $\sigma \in \Sigma$, the orientation obtained by applying the reduction rules as far as possible is said to be* the reduced form of σ. *Let p and p' be* scurves*, also let $\sigma = orn(p)$ and $\sigma' = orn(p')$. If σ' is the reduced form of σ, then p' is said to be* the reduced form of p.

For example, $\sigma_1 = +\underline{++--}++-$ is reduced to $\sigma_2 = ++-++-$ by applying (r2) to the underlined part. Then $\sigma_2 = +\underline{+-+}+-$ is reduced to $\sigma_3 = +++-$ by applying (r1) to the underlined part. σ_3 is the reduced form of σ_1.

The following properties hold regarding the reduction.

Proposition 1. *1. (termination)*
The reduction procedure terminates.
2. (rotation difference preservation)
The rotation difference is preserved in the reduction.
3. (reduced form)
(a) The reduced form is a nonempty sequence of the same symbol enclosed by at most one opposite symbol on each end.
(b) The symbols at both ends are not changed in the reduction.
4. (confluence)
There is a unique reduced form, regardless of the order of rule application.

Proof)

1.2. They are trivial, since a given sequence is finite and the segments are eliminated by a pair of + and −, or two pairs of them.

3(a). Let $s \in \{+, -\}$ and $\sigma' = \sigma_1 \sigma_2 \sigma_3$ be the reduced form where σ_2 is a nonempty sequence of s. Assume that σ_1 or σ_3 is a sequence of more than one $inv(s)$. Without a loss of generality, assume that $\sigma_1 = --$. Then there are two cases: if $\sigma' = --++\ldots$, then (r2) is applicable, and if $\sigma' = --+-\ldots$, then (r1) is applicable; both contradict the definition of the reduced form. Therefore, both of σ_1 and σ_3 should be sequences of at most one $inv(s)$.

3(b). It clearly holds from the definition of the reduction rules.

4. Consider that there exist more than one subsequence to which a rule is applicable. If the subsequences do not overlap, it does not matter whichever rule is applied first. Therefore, we consider the cases in which the subsequences overlap. Due to the symmetricity of + and −, and that of the order of the sequence, it is sufficient to consider only three patterns as such cases: $+-+-$, $++--+-$ and $++--++$. The reduced forms of these sequences are $+-$, $+-$ and $++$, respectively, regardless of the order of rule applications. Therefore, the confluence holds. □

In addition to these properties, reduction process preserves admissibility. The following Lemma 1 and Lemma 2 present that (r1) and (r2) preserve admissibility, respectively. Their proofs are based on the qualitative treatment of segments: segments with the same abstract direction and curvature are embeddings of the same qualitative representation. We just show the proof idea, due to the page limit. The detail of the proofs is shown in the appendix.

Lemma 1. *Let an* scurve $p = X_1 \cdot \ldots \cdot X_n$ *(*$n \geq 3$*). Assume that* $orn(X_{i-1}) = inv(orn(X_i)) = orn(X_{i+1})$ *(*$\forall i; 2 \leq i \leq n-1$*). If we take* X' *such that* $X' = X_{i-1}$ *holds, then* $p' = X_1 \cdot \ldots \cdot X_{i-2} \cdot X' \cdot X_{i+2} \cdot \ldots \cdot X_n$ *is also an* scurve *from* X_1 *to* X_n*. Moreover,* p' *is admissible if and only if* p *is admissible.*

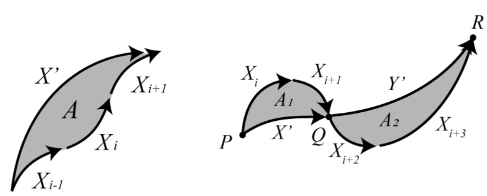
Proof) In case $i = 2$ or $i = n-1$, we do not care the direct connection of X' and the adjacent segments after the rule is applied. If $i \neq 2$ then $dc(X_{i-2}, X')$, and if $i \neq n-1$ then $dc(X', X_{i+2})$, from $X' = X_{i-1} \wedge X' = X_{i+1}$. Therefore, p' is also an *scurve* from X_1 to X_n.

Next, assume that p is admissible. As there exists an open embedding of p, we draw $\boldsymbol{X}'$ in this embedding in a manner such that $init(\boldsymbol{X}') = init(\boldsymbol{X}_{i-1})$ and $term(\boldsymbol{X}') = term(\boldsymbol{X}_{i+1})$ are satisfied (Figure 5(a)). Let A be a region that is enclosed by $\boldsymbol{X}_{i-1}, \boldsymbol{X}_i, \boldsymbol{X}_{i+1}$ and $\boldsymbol{X}'$. We may move or stretch/shrink the embedding of the segments with keeping their connections, so that neither of the parts $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_{i-1}$ nor $\boldsymbol{X}_{i+1} \cdot \ldots \cdot \boldsymbol{X}_n$ intersects with the region A. Then, we get an open embedding of p'. Therefore, p' is admissible.

Conversely, assume that p' is admissible. Take an open embedding of p'. For this embedding, we can draw $\boldsymbol{X}_{i-1}, \boldsymbol{X}_i$ and $\boldsymbol{X}_{i+1}$ so that their curvatures are sufficiently small. Note that there always exists a space in which these segments can be drawn. Then it is an open embedding of p. □

Lemma 2. *Let* $p = X_1 \cdot \ldots \cdot X_n$ *(*$n \geq 4$*). Assume that* $orn(X_i) = orn(X_{i+1}) = inv(orn(X_{i+2})) = inv(orn(X_{i+3}))$ *(*$\forall i; 1 \leq i \leq n-3$*). If we take* X', Y' *such that* $X' = X_i$ *and* $Y' = X_{i+3}$ *hold, then* $p' = X_1 \cdot \ldots \cdot X_{i-1} \cdot X' \cdot Y' \cdot X_{i+4} \cdot \ldots \cdot X_n$ *is also an* scurve *from* X_1 *to* X_n*. Moreover,* p' *is admissible if and only if* p *is admissible.*

Proof) The relative positional relations among the segments in open embeddings of p and p' are shown in Figure 5(b). This lemma is proved similarly with Lemma 1, by considering regions A_1 and A_2 and splitting the cases by the relative positional relations among the points P, Q and R. □



(a) r1 (b) r2

Fig. 5. Admissibility preservation.

From Lemma 1 and Lemma 2, we have gotten the following theorem.

Theorem 1. *An* scurve *is admissible iff its reduced form is admissible.*

4.3 Admissibility of an *scurve*

We show the admissibility of an *scurve* p.

Let n be the length of p. There are 2^n patterns of its orientation in total. Although this number is decreased to 2^{n-2} by considering the symmetricity of $+$ and $-$, and that of the order of the sequence, it is still large. The introduction of the reduction significantly decreases the number of *scurve*s to be checked, since the length is shortened and the reduced forms are restricted. Let k be rotation difference of p. When $n \geq 8$, it is not admissible since $k \geq 4$. When $n \leq 7$, the number of the cases can be decreased to only 11 patterns by the reduction. We show the result of the investigation of these 11 patterns in Figure 6. As a result, two patterns in which $k = 3$ are not admissible.

From above all, we conclude that for any *scurve*, its admissibility can be determined by its orientation, and we have gotten the following theorem.

Theorem 2. *Let n be the length of a given* scurve *p, and k be its rotation difference. When $n \leq 3$, p is admissible. When $n \geq 4$, p is admissible iff $k < 3$.*

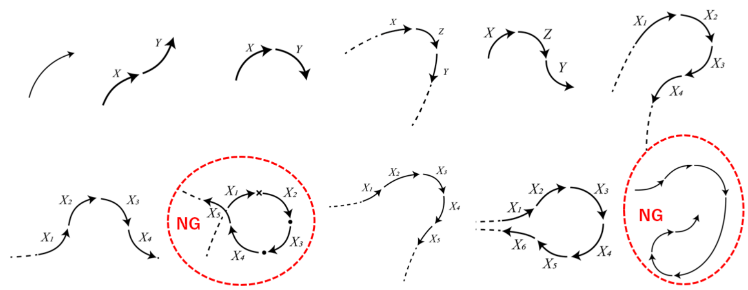
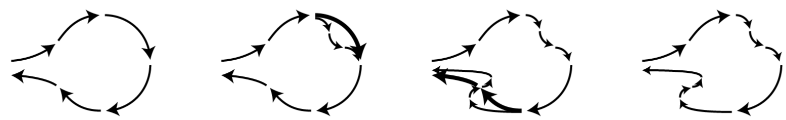


Fig. 6. Embeddings of all possible patterns.

4.4 Example of drawing

When an admissible *scurve* p is given, we can actually draw a figure that does not form a spiral on a two-dimensional plane as follows: take the figure in Figure 6 which corresponds to its reduced form, and replace the segments in the manner based on the proofs of Lemma 1 and Lemma 2.

For example, consider an *scurve* p where $orn(p) = -++-++++--$. p is reduced to p_1 where $orn(p_1) = -++-+++-$ by (r2). p_1 is reduced to p_2 where $orn(p_2) = -++++-$ by (r1), which is the reduced form of p. p_2 is admissible and can be drawn as a figure shown in Figure 7(a). p_1 can be drawn by replacing the third segment of p_2 (Figure 7(b)). p can be drawn by replacing the seventh and eighth segments of p_1 (Figure 7(c)). Finally, we have gotten the curve shown in Figure 7(d) which is an open embedding of p.



(a) p_2 (b) $p_2 \Rightarrow p_1$ (c) $p_1 \Rightarrow p$ (d) p

Fig. 7. Example of drawing an *scurve*.

5 Related Works

Although there have been lots of research on QSR, few of them focus on shapes, especially on curves. Several systems in these works divide the boundary of an object into segments and represent its shape by lining up the symbols corresponding to the segments. Leyton proposed the division of a closed curve into segments based on convexity and defined the grammar based on the symbols corresponding to the primitive segments [9]. He also proposed the representation of the change of the shapes of closed curves, using this grammar. Tosue et al. extended this grammar so that tangent points and division of closed curves can be handled [15]. Galton et al. defined the grammar that can treat not only curved segments but also straight segments and cusps [5]. Cabedo et al. proposed the description in which each segment is equipped with additional information such as relative length and angles and extended it so that it may treat juxtaposition of objects [1, 11, 4]. Kulik et al. applied QSR to landscape silhouettes [8]; they proposed a descriptive language for the shape of an open line that is the border of a landscape seen from the horizontal perspective. However, these existing systems do not treat the connection between objects in a distant location.

6 Conclusion

We have shown a revised judgment method for determining the spiral form and demonstrated that it is possible to generate a curve with an abstract shape from the specified expression, when it is judged as not forming a spiral.

Curves are commonly observed in many natural objects, ranging from microscopic structures like cells to macroscopic features such as terrains. Frequently, we encounter situations where it is necessary to predict the entire shape of a curve from partially disclosed or unclear data. The method proposed in this paper offers a novel approach for reasoning about spatial data with uncertainty, serving as an alternative to machine learning-based methods. It may have applications in areas such as medical image processing, where data often include ambiguous or missing parts. An application for predicting the abstract shape of a global geological stratum from a set of local strata data was suggested in our previous paper [13]. The condition of an *scurve* is relevant to wiring problems in circuit design.

In future, we aim to develop a new QSR calculus to address the connecting relationships between curved segments.

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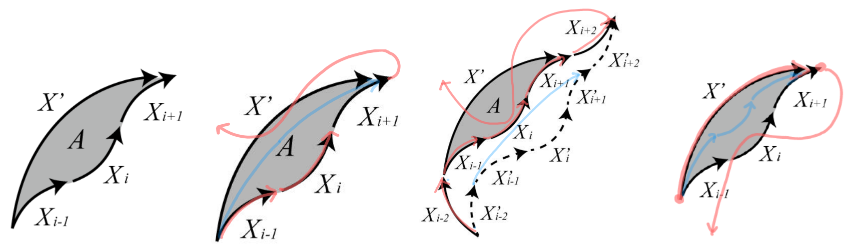
Appendix

Proof of Lemma 1)

In case $i = 2$ or $i = n - 1$, we do not care the direct connection of X' and the adjacent segments after the rule is applied, since the part to which the rule is applied is one of the ends of an *scurve*. If $i \neq 2$ then $dc(X_{i-2}, X')$, and if

$i \neq n-1$ then $dc(X', X_{i+2})$, from $X' = X_{i-1} \wedge X' = X_{i+1}$. Therefore, p' is also an *scurve* from X_1 to X_n.

Next, assume that p is admissible. As there exists an open embedding of p, we can draw $\boldsymbol{X}'$ in this embedding in a manner such that $init(\boldsymbol{X}') = init(\boldsymbol{X}_{i-1})$ and $term(\boldsymbol{X}') = term(\boldsymbol{X}_{i+1})$ are satisfied (Figure 8(a)). Let A be a region that is enclosed by $\boldsymbol{X}_{i-1}, \boldsymbol{X}_i, \boldsymbol{X}_{i+1}$ and $\boldsymbol{X}'$.



(a) (b) (c) (d)

Fig. 8. Admissibility preservation of (r1).

In this process, as Figure 8(b) shows, if we can draw $\boldsymbol{X}'$ (shown in the blue line) with sufficiently small curvature so that neither of the parts $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_{i-1}$ nor $\boldsymbol{X}_{i+1} \cdot \ldots \cdot \boldsymbol{X}_n$ (shown in the red line) intersects with the region A, then this is an open embedding of p'. However, as Figure 8(c) shows, if we cannot draw such an $\boldsymbol{X}'$, then we modify this embedding by moving the part $\boldsymbol{X}_{i-2} \cdot \boldsymbol{X}_{i-1} \cdot \boldsymbol{X}_i \cdot \boldsymbol{X}_{i+1} \cdot \boldsymbol{X}_{i+2}$ in the lower direction with the locations of $init(\boldsymbol{X}_{i-2})$ and $term(\boldsymbol{X}_{i+2})$ fixed, to get $\boldsymbol{X}'_{i-2} \cdot \boldsymbol{X}'_{i-1} \cdot \boldsymbol{X}'_i \cdot \boldsymbol{X}'_{i+1} \cdot \boldsymbol{X}'_{i+2}$, shown by the dotted line in the figure [1]. The resulting curve is also an open embedding of p. Then, we draw $\boldsymbol{X}'$ with sufficiently small curvature so that the new enclosed region is sufficiently thin. Then, this is an open embedding of p'.

Conversely, assume that p' is admissible (Figure 8(d)). Take an open embedding of p' (red line). For this embedding, we can draw $\boldsymbol{X}_{i-1}, \boldsymbol{X}_i$ and $\boldsymbol{X}_{i+1}$ so that their curvatures are sufficiently small (blue line). Note that there always exists a space in which these segments can be drawn. Then it is an open embedding of p. □

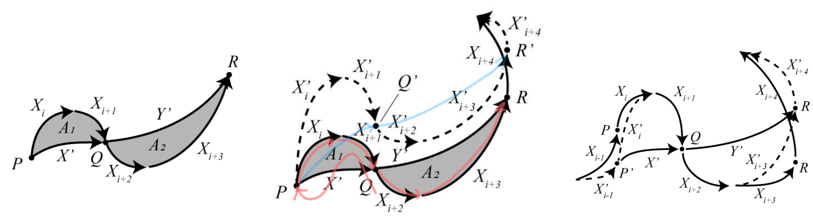
Proof of Lemma 2)

In case $i = 1$ or $i = n-3$, we do not care the direct connection of X', Y' and the adjacent segments after the rule is applied, since the part to which the rule is applied is one of the ends of an *scurve*. If $i \neq 1$ then $dc(X_{i-1}, X')$, and if

[1] We show one representative case. The moving direction is not always the lower one, but depends on the $dir(\boldsymbol{X}_i)$. In addition, other segments should be moved together in some cases. These other cases can be handled in a similar manner.

$i \neq n-3$ then $dc(X', X_{i+4})$, from $X' = X_{i-1} \wedge Y' = X_{i+3}$. Moreover, $dc(X', Y')$ holds. Therefore, p' is an *scurve* from X_1 to X_n.

Next, assume that p is admissible. As there exists an open embedding of p, we can draw $\boldsymbol{X}'$ in this embedding in a manner such that $init(\boldsymbol{X}') = init(\boldsymbol{X}_i)$ and $term(\boldsymbol{X}') = term(\boldsymbol{X}_{i+1})$ are satisfied, and $\boldsymbol{Y}'$ in a manner $init(\boldsymbol{Y}') = init(\boldsymbol{X}_{i+2})$ and $term(\boldsymbol{Y}') = term(\boldsymbol{X}_{i+3})$ are satisfied (Figure 9(a)). Let A_1 be the region enclosed by $\boldsymbol{X}_i, \boldsymbol{X}_{i+1}$ and $\boldsymbol{X}'$, and A_2 be the region enclosed by $\boldsymbol{X}_{i+2}, \boldsymbol{X}_{i+3}$ and $\boldsymbol{Y}'$. Also let P, Q and R be $init(\boldsymbol{X}_i), init(\boldsymbol{X}_{i+2})$ and $term(\boldsymbol{X}_{i+3})$, respectively.



(a) (b) (c)

Fig. 9. Admissibility preservation of (r2).

(Case 1) $dir(\boldsymbol{X}_i \cdot \boldsymbol{X}_{i+1}) = dir(\boldsymbol{X}_{i+2} \cdot \boldsymbol{X}_{i+3}) = dir(\boldsymbol{X}_i)$.

If we can draw $\boldsymbol{X}'$ with sufficiently small curvature so that neither of the parts $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_{i-1}$ nor $\boldsymbol{X}_{i+4} \cdot \ldots \cdot \boldsymbol{X}_n$ intersects with A_1 nor A_2. Then we get an open embedding of p'.

Otherwise, as Figure 9(b) shows, if either of the parts $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_{i-1}$ or $\boldsymbol{X}_{i+4} \cdot \ldots \cdot \boldsymbol{X}_n$ intersects with A_1 or A_2 (red line), then we modify this embedding by moving Q and R in the upper direction to Q' and R', respectively, and draw $\boldsymbol{X}'_i, \boldsymbol{X}'_{i+1}, \boldsymbol{X}'_{i+2}$ and $\boldsymbol{X}'_{i+3}$ shown by the dotted line in this figure. As a result, neither of the parts $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_{i-1}$ nor $\boldsymbol{X}_{i+4} \cdot \ldots \cdot \boldsymbol{X}_n$ intersects with A_1 nor A_2. The resulting curve is also an open embedding of p. Then, we draw $\boldsymbol{X}'$and $\boldsymbol{Y}'$ (blue line). Therefore, we can get an open embedding of p'

(Case 2) $dir(\boldsymbol{X}_i \cdot \boldsymbol{X}_{i+1}) \neq dir(\boldsymbol{X}_i)$ or $dir(\boldsymbol{X}_{i+2} \cdot \boldsymbol{X}_{i+3}) \neq dir(\boldsymbol{X}_i)$.

We modify the embedding of p by moving P to P' in the lower direction and R to R' in the upper direction with the locations of $init(\boldsymbol{X}_{i-1})$ and $term(\boldsymbol{X}_{i+4})$ fixed, so that $dir(\boldsymbol{X}_i \cdot \boldsymbol{X}_{i+1}) = dir(\boldsymbol{X}_{i+2} \cdot \boldsymbol{X}_{i+3}) = dir(\boldsymbol{X}_i)$ is satisfied to get $\boldsymbol{X}'_{i-1}, \boldsymbol{X}'_i, \boldsymbol{X}'_{i+3}, \boldsymbol{X}'_{i+4}$ shown by the dotted line (Figure 9(c)). The relative relations of P', Q and R' are the same as that in (Case 1). Therefore, we can get an open embedding of p'.

Conversely, assume that p' is admissible. Take an open embedding of p'. For this embedding, we can draw $\boldsymbol{X}_i, \boldsymbol{X}_{i+1}, \boldsymbol{X}_{i+2}$ and $\boldsymbol{X}_{i+3}$ so that their curvatures are sufficiently small. Then it is an open embedding of p. □