Operations for Shape Transformations Based on Angles

Momo Tosue[1], Sosuke Moriguchi[1] and Kazuko Takahashi[1]
[1] *Kwansei Gakuin University, 2-1 Gakuen, Sanda, Hyogo, Japan*
eyi98098@kwansei.ac.jp, chiguri@acm.org, ktaka@kwansei.ac.jp

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Abstract: We propose a symbolic expression for a qualitative shape of an object in the sequence of rotation angles of edges. We give a drawing algorithm for the expression based on rewriting strings and prove that we can draw a figure on a two-dimensional plane, for a consistent expression. We also refine this algorithm as an abstract rewriting system to represent shapes of figures and their changes, and prove that the system has confluence and termination.

1 INTRODUCTION

Work in the field of artificial intelligence involves symbolic treatment of spatial data. When we look at images or diagrams, it is easier to recognize the features thereof (shape, relative position, and relative size). However, when we discuss the properties of such features, it is appropriate to express the figure symbolically. Symbolic treatment creates a representation amenable to human understanding and enables the use of computational tools in the formalization, whereas numerical data cannot be treated in this manner.

Qualitative Spatial Temporal Reasoning is the subfield of knowledge representation and symbolic reasoning (Ligozat, 2011). It focuses on certain aspects of an object, and reasons by reference to those aspects, without using precise numerical data. Application areas including robot navigation and geographic information systems have been proposed. Both shapes and their transformations are considered as important aspects in these applications, and several studies have explored the symbolic representation of object shapes in the two-dimensional plane. Most represented object shapes by tracing the boundaries (Leyton, 1988; Galton and Meathrel, 1999; Museros and Escrig, 2004; Schlieder, 1996; Kulik and Egenhofer, 2003; Gottfried, 2003; Gottfried, 2004). A boundary is a closed figure usually divided into segments. The boundary is thus represented as a sequence of segments, sometimes combined with relationships between adjacent segments. The shape features distinguish straight from curved lines, the sizes of edges and angles. Cohn took a different approach in which an outline is not traced to represent a qualitative shape of an object (Cohn, 1995). He developed a representation using relationships over regions. Convexity was considered, with a hierarchical focus on the difference between the region occupied by a given object and its convex hull. Tosue et al. developed a symbolic expression incorporating concavity, tangent points, and divisions of an object, as well as a transition system reflecting changes in shape, aiming at a qualitative simulation and backward reasoning on the organogenesis process (Tosue et al., 2018).

These studies considered that the shape of a figure in the two-dimensional plane could be represented using the expressions developed, but did not discuss the opposite proposition, that is, the existence of a figure corresponding to a given expression. Also, if there exists a figure, the algorithm to draw a figure was not shown. Especially, it is difficult to define a sequence of operations that draws a closed figure with concave parts because unlike the situation when dealing with figures with convex regions, it is necessary to locate the vertices in positions ensuring that edges do not intersect.

In this paper, we modify the qualitative expression proposed in (Tosue et al., 2018) to explore this problem. Intuitively, the expression is the trace of the outline of an object. We approximate a boundary as a polygon and determine the direction of each edge, starting at an arbitrary vertex and tracing the boundary counter-clockwise (as viewed from the left). We consider a finite sequence of directed edges with $\pi/3$ steps, the lengths of which are ignored. We define the angle of rotation between adjacent directed edges. When the vertex is convex, the angle is positive; when the vertex is concave, the angle is negative. The expression is a sequence of the angles of the rotations.

For example, the figures in Figure 1(a),(b) and (c) are expressed as the following sequences: $(2\pi/3, 2\pi/3, 2\pi/3)$, $(2\pi/3, 2\pi/3, -2\pi/3, 2\pi/3, 2\pi/3)$, and $(2\pi/3, 2\pi/3, 2\pi/3, -\pi/3, \pi/3)$, respectively. In the figure, the bold arrow shows the startpoint of the sequence.

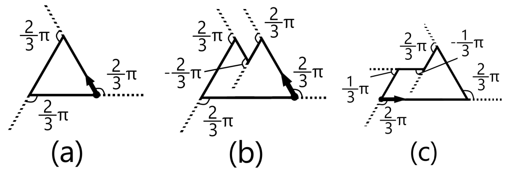


Figure 1: Angles of rotation.

It is known that the sum of the angles of rotation is 2π if and only if a closed figure can be drawn without an intersection on a two-dimensional plane. Tosue et al. required this proposition to define a consistency, which is the condition to be satisfied by an expression for the existence of a corresponding figure. However, they did not show an algorithm that drew the figure. In this paper, we present an algorithm that draws a figure corresponding to the expression and present a constructive proof of the algorithm.

We formalize the expression and the algorithm as an abstract rewriting system (Klop, 1992). An abstract rewriting system is generally used for discussing computational models, but can be employed more widely to formalize a rewriting system. Here, we define an abstract rewriting system over a set of expressions and the rewriting rules, and present a constructive proof. Each rewriting rule corresponds to the deletion of a concave part. Starting with a figure corresponding to a consistent expression, the concave parts are deleted one-by-one in each step until a simple convex shape remains. The inverse of this procedure yields an algorithm that draws a figure corresponding to a given consistent expression starting from a simple convex shape.

It is crucial to define the rewriting rules. If we try to rewrite an edge, many possible shapes may be obtained starting with a single shape. For example, consider the figures shown in Figure 2. Figure (a) can be obtained from all shapes (b)-(f) (and more) by rewriting one or two edges. It is thus essential to define rules embracing all possible cases, but this is burdensome. Here, we define rules for rewriting angles instead of edges. Then, we require only four rules, which greatly simplifies our proof.

Using this approach, the following questions arise: (1) When a shape changes to another shape, is there more than one sequence of rewriting steps? (2)

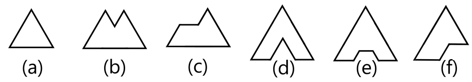


Figure 2: Rewriting rules.

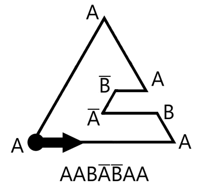


Figure 3: Example with its corresponding expression.

For a given shape with concave parts, does the rewriting terminate, and is the given shape always rewritten to the same convex shape?

These two questions are reduced to the principal issues addressed by abstract rewriting systems: confluence and termination. In this paper, we show that when a shape changes to another shape, there may be more than one sequence of rewriting steps, and for a given shape with concave parts, rewriting terminates at the same convex shape.

The remainder of this paper is organized as follows. In Section 2, we define our descriptive language and rewriting system. In Section 3, we describe the algorithm used to draw a figure. In Section 4, we prove that the system exhibits confluence and termination. In Section 5, we discuss the properties of the system. In Section 6, we provide conclusions and describe our planned future work.

2 LANGUAGE

Here, we define our directional language $\mathcal{D}$. As explained above, $\mathcal{D}$ denotes figures as sequences of rotational angles. We use A, B, $\overline{A}$, and $\overline{B}$ to denote rotations of $2\pi/3$, $\pi/3$, $-2\pi/3$, and $-\pi/3$, respectively. For example, figure (a) in Figure 1 is described by AAA. Figures (b) and (c) are described by $AA\overline{A}AA$ and $AAA\overline{B}\overline{B}$, respectively. Another example is shown in Figure 3; the figure is described by $AAB\overline{A}\overline{B}AA$.

In $\mathcal{D}$, any expression denotes a closed figure. The startpoint of the rotational sequence is irrelevant. The use of different startpoints simply rotates the expression; for example $AB\overline{A}\overline{B}AAA$ or $B\overline{A}\overline{B}AAAA$ in Figure 3. These expressions correspond to the same figure.

The formal definition of $\mathcal{D}$ is as follows:

Definition 1 (Expression)**.** *An expression in $\mathcal{D}$ is defined as a finite sequence of* $\{A, B, \overline{A}, \overline{B}\}$ *(i.e.,* $x_1 \ldots x_n$

where $x_i \in \{A, B, \overline{A}, \overline{B}\}$ for all i). We use ε to denote the empty sequence, and uv the concatenation of two sequences u and v.

Definition 2 (Rotation)**.** *We term the function r defined as $r(A) = 2\pi/3$, $r(B) = \pi/3$, $r(\overline{A}) = -2\pi/3$, and $r(\overline{B}) = -\pi/3$, the rotation function. For the sequence $u = x_1 \dots x_n$, we write $r(u)$ as $\Sigma_{i=1}^{n} r(x_i)$.*

In terms of the signs returned by the rotation function, we consider A and B (i.e., counter-clockwise rotations) to be *positive* and $\overline{A}$ and $\overline{B}$ (clockwise rotations) to be *negative* rotations. Hereafter, we use x and y for A, B, $\overline{A}$ or $\overline{B}$, and u, v and w for sequences of them.

Certain sequences of rotation angles cannot be used to create closed figures. For example, BAB should be a triangle (because it has three angles), but the sum of the inner angles is not π, thus contradicting an essential feature of triangles. We use the following condition to denote (only) closed figures.

Definition 3 (Consistency)**.** *Let e be an expression of $\mathcal{D}$. If $r(e) = 2\pi$, the expression is consistent.*

For the expression in Figure 3, $r(AAB\overline{A}\overline{B}AA) = r(A) * 4 + r(B) + r(\overline{A}) + r(\overline{B}) = 2\pi/3 * 4 + \pi/3 - 2\pi/3 - \pi/3 = 2\pi$. This satisfies the consistency condition. However, a question arises: Does this guarantee the existence of figures corresponding to the given expression? We answer this question in the next section.

3 REALIZABILITY

In this section, we offer a constructive proof of realizability; we develop a drawing algorithm for the two-dimensional plane. Rewriting in $\mathcal{D}$ corresponds to an operation on the figure. We consider the operation featuring addition of a convex part rather than deletion of a concave part.

3.1 Smoothing Regions

Fundamentally, the drawing algorithm introduced in this section seeks to smooth the smoothable regions of a figure. Here, a "smoothable region" means a pair of adjacent rotation angles, of which the first is negative and the second positive[1]. We write the expression $x_1 \dots x_i x_{i+1} \dots x_n$ has a smoothable region at i if x_i is either $\overline{A}$ or $\overline{B}$ and x_{i+1} is either A or B. In addition, if x_n is either $\overline{A}$ or $\overline{B}$ and x_1 is either A or B, we write that the expression exhibits a smoothable region at n, and, in this case, that region is $x_n x_1$.

[1]It is possible to define a smoothable region as a pair, of which the first is positive and the second is negative. The characteristics can be discussed similarly, except that smoothing can be considered as filling of concave parts instead of excision.

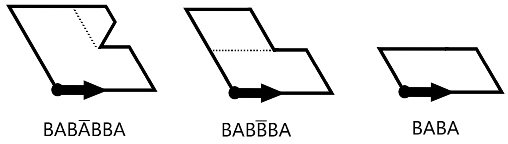


Figure 4: Smoothing of a figure.

Definition 4 (Smoothing)**.** *Smoothing is an operation in which a smoothable region in an expression is rewritten to become zero or one rotation angle, following certain rules:*

1. *$\overline{B}A$ is replaced by B.*
2. *$\overline{A}B$ is replaced by $\overline{B}$.*
3. *$\overline{A}A$ is removed (replaced by ε).*
4. *$\overline{B}B$ is removed (replaced by ε).*

After smoothing, the expression e' obtained is termed the smoothed expression of e.

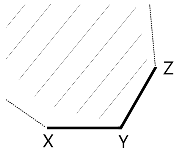
For example, $BAB\overline{A}BBA$ is rewritten to $BAB\overline{B}BA$, and then to $BABA$. This process is shown in Figure 4. Intuitively, the operation features "excision" of the smoothable region (the dotted line) and smoothing of the figure.

This operation enables us to draw a figure corresponding to a given expression using the figure corresponding to the smoothed expression. For example, Figure 4 shows that if $BABA$ is available, we can draw $BAB\overline{B}BA$ and $BAB\overline{A}BBA$. We now prove this. For the brevity, we skip some cases in the proof here; see Appendix for the skipped cases.

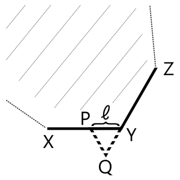
Theorem 1. *Let e be a consistent expression and e' be the smoothed expression of e. If e' is drawable in the two-dimensional plane without crossing, then so is e.*

Proof. Assume that F is the figure corresponding to e'. Let d be the minimum of the distances between two non-adjacent edges in F. From the definition, d is less than the length of any edge. We draw the additional edges in the region within d from F, which makes these edges not intersect with F. We separate the methods used to smooth e.

Case 1: When $\overline{B}A$ is replaced by B. In F, this B is drawn as points X, Y, and Z, as shown in the following figure. Note that in the following figures, the shaded areas indicate the inside.

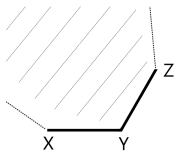


Let l be less than d. On XY, choose a point P so that the length of PY is l. Also, choose a point Q so that PQY is a regular triangle and Q, Y, and Z lie on the same line. As l is less than d, PQ and QY do not intersect with the other edges of F. The region $XPQZ$ corresponds to $\overline{B}A$, and the rest is the same as e'. This means that the figure is denoted by e.

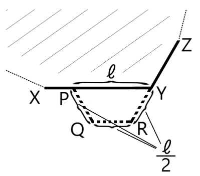


Case 2: When $\overline{A}B$ is replaced by $\overline{B}$. Skip.
Case 3: When $\overline{A}A$ is replaced by ε. Skip.
Case 4: When $\overline{B}B$ is replaced by ε. Assume that x is the next rotation angle of $\overline{B}B$ in e. We divide the case depending on the value of x.

- $x = B$: In F, x is drawn as shown below.



Let l be less than d. On XY, we place a point P so that the length of PY is l. We create the isosceles trapezoid $PQRY$; the lengths of PQ, RY and QR are $l/2$. Then, the distances between any points on QR and the edge XY are less than d; thus, QR also does not intersect with any other edge in F. The region $XPQRZ$ corresponds to $\overline{B}BB$, and the rest is the same as e'. This means that the figure is denoted by e.



x=B

- The other cases: Skip.

□

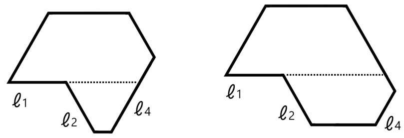


Figure 5: The applicable figure (left) and the inapplicable figure (right) of the smoothing operation.

Note that some figures are not amenable to smoothing. For example, the two figures in Figure 5 correspond to the same expression, $A\overline{B}BBBBB$, which is smoothed to $ABBBB$. On the figure, this smoothing should be done in a manner that we connect edge l_1 to edge l_4 by extending l_1, as in the left figure. However, in the right figure, l_4 is too short to connect to l_1. The drawing in the above proof reveals several limitations of the results. For example, when $\overline{B}BB$ is processed as a part of e (see the figure in the proof in Case 4), PQ is always shorter than RZ. This is not true in the right figure of Figure 5; the drawing never generates this figure.

The smoothing operation simplifies the expression, rendering it easier to draw in the two-dimensional plane. To apply the operation repeatedly, consistency of the expression should be preserved.

Theorem 2. *Let e be a consistent expression. If e has a smoothable region, then the smoothed expression is also consistent.*

Proof. Assume that a smoothable region at i exists in e. In such a case, we can write e as $ux_ix_{i+1}v$ where $u = x_1 \dots x_{i-1}$ and $v = x_{i+2} \dots x_n$. When x_ix_{i+1} is replaced by w during smoothing, the smoothed expression is described as uwv.

As e is consistent, $r(e) = 2\pi$. From the definition of the rotation function, $r(e) = r(ux_ix_{i+1}v) = r(u) + r(x_i) + r(x_{i+1}) + r(v)$. For every smoothing rule, it is easy to check that $r(x_i) + r(x_{i+1}) = r(w)$. Thus $r(uwv) = r(u) + r(w) + r(v) = r(u) + r(x_i) + r(x_{i+1}) + r(v) = r(e) = 2\pi$ (i.e., the smoothed expression is consistent). □

3.2 Normal Forms

We term an expression that cannot be smoothed a *normal form* of the expression. We prove that there are only five normal forms in $\mathcal{D}$.

Theorem 3. *If a consistent expression is a normal form (i.e., has no smoothable region), it is one of BBBBBB, ABBBB, ABAB, AABB, or AAA, or a rotation thereof.*

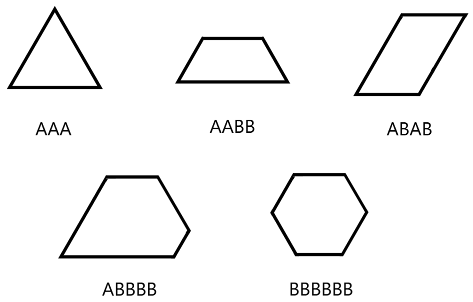


Figure 6: The normal forms.

Proof. Consistent expressions should have positive rotations because the sum of the rotations is positive (2π). If negative rotations exist, then at least one such rotation lies adjacent to a positive rotation, which means that a smoothable region exists. Therefore, if e cannot be smoothed, e features only positive rotations. Positive rotations are either $\pi/3$ or $2\pi/3$, and the numbers of B and A are thus $(6,0)$, $(4,1)$, $(2,2)$, or $(0,3)$. Clearly, all of these expressions are either of the form *BBBBBB*, *ABBBB*, *ABAB*, *AABB*, or *AAA*, or a rotation thereof. □

Figure 6 shows the five normal forms and this shows that all normal forms are drawable in the two-dimensional plane without crossing.

The last step when drawing a figure is to ensure that the operation creates a normal form from the given expression.

Theorem 4. *For any consistent expression e, e becomes a normal form after a finite number of smoothings.*

Proof. The smoothing operation decreases the length of any given expression; thus, after a finite number of steps, the expression becomes an expression that cannot be further smoothed (i.e., a normal form). □

3.3 Drawing Algorithm

The following is the algorithm used to draw consistent expressions. The correctness of the algorithm is assured by reference to Theorem 1, 3 and 4.

This algorithm shows that, for any consistent expression, there are figures in the two-dimensional plane corresponding to the expression.

We show drawing of a figure $AAB\bar{A}\bar{B}AA$ shown in Figure 3. First, we find its normal form. The expression is smoothed to $AAB\bar{A}BA$, $AAB\bar{B}A$, and then to $AABB$, a normal form.

Next, starting from a normal form $AABB$, we draw a figure for $AAB\bar{A}\bar{B}AA$. The process is shown in Figure 7. $AABB$ is drawn as the trapezoid (a) in the figure. As $AABB$ is a smoothed expression of $AAB\bar{B}A$, we place a regular triangle in the top left of the trapezoid ((b) in the figure). Next, we place a parallelogram on the triangle ((c) in the figure), as the parallelogram is a smoothable region of $AAB\bar{A}BA$. Finally, we obtain a figure corresponding to $AAB\bar{A}\bar{B}AA$ ((d) in the figure).

Algorithm 1 Drawing Expression

while smoothable region is located at *i* **do**
  Push *i* and the shape of the smoothable region to stack
  Smooth smoothable region at *i*
end while
Draw normal form
while stack is not empty **do**
  Pop the information about smoothable region
  Reconstruct the smoothable region
end while

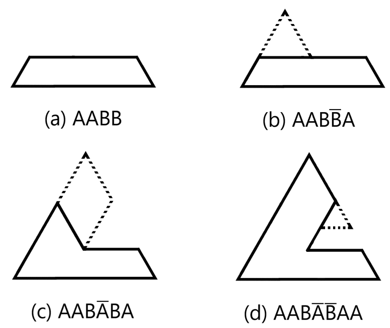


Figure 7: Drawing of the expression in Figure 3.

We have implemented a prototype system based on this algorithm [2].

4 REWRITING SYSTEM

We refine the algorithm defined in the previous section by creating a rewriting system. We use the alphabet set $\Sigma = \{A, B, \bar{A}, \bar{B}\}$ and the relation $R = \{(\bar{A}A, \varepsilon), (\bar{B}B, \varepsilon), (\bar{A}B, \bar{B}), (\bar{B}A, B)\}$.

Definition 5. *$\mathcal{S} = (T, \to)$ is a rewriting system where $T = \Sigma^*$ and $\to = \{(u\bar{x}yv, uwv) | u\bar{x}yv \in T \wedge (\bar{x}y, w) \in R\} \cup \{(yu\bar{x}, wu) | yu\bar{x} \in T \wedge (\bar{x}y, w) \in R\}$.*

The latter set in the definition of $\to$ enables us to rewrite the smoothable region at the tail of expressions. This is an instance of cycle rewriting of string

[2] https://ist.ksc.kwansei.ac.jp/~ktaka/QSRDrawer/RotationApplication.jar

rewriting systems (Zantema et al., 2014) with the alphabet set Σ and the relation R.

We denote $e \to e'$ when $(e, e') \in \to$. Also, we use $\to^*$ as the reflexive-transitive closure of $\to$.

This system has several useful properties. Here, we prove the confluence of rewriting and the presence of a uniquely normalizing property.

Theorem 5. *The rewriting system $\mathcal{S}$ exhibits confluence. In other words, for any expression $e \in T$, if there exist two expressions e_1 and e_2 such that $e \to^* e_1$ and $e \to^* e_2$, then there exists e_3 such that $e_1 \to^* e_3$ and $e_2 \to^* e_3$*

Proof. The general confluence property is derived from the one-step confluence property (i.e., if $e \to e_1$ and $e \to e_2$, then $e_1 = e_2$, or $e_1 \to e_3$ and $e_2 \to e_3$). From the definition of R, the regions smoothed during the transition of e to e_1 and e to e_2 are either the same or non-overlapping. If they are the same, clearly $e_1 = e_2$. Otherwise, we can write $e = u\overline{x_1}y_1v\overline{x_2}y_2w$, $e_1 = uw_1v\overline{x_2}y_2w$, and $e_2 = u\overline{x_1}y_1vw_2w$, where $(\overline{x_1}y_1, w_1), (\overline{x_2}y_2, w_2) \in R$. Then, it is clear that $e_3 = uw_1vw_2w$ and $e_1 \to e_3$ and $e_2 \to e_3$. Therefore, the system exhibits one-step confluence, and is thus confluent. □

The confluence property is used to derive the following theorem.

Theorem 6. *The rewriting system $\mathcal{S}$ exhibits a uniquely normalizing property. In other words, for any expression $e \in T$, there is at most one normal form e' such that $e \to^* e'$.*

The system $\mathcal{S}$ exhibits a termination property, as revealed by Theorem 4. This means that every expression e has exactly one normal form. We discuss classification based on normal forms in Section 5.2.

5 DISCUSSION

We discuss some properties of our rewriting system.

5.1 Symmetric Expression

Definition 6 (Symmetric). *Let e_1 and e_2 be expressions in $\mathcal{D}$. If $e_1 = x_1 \ldots x_n$ and $e_2 = x_n \ldots x_1$, they are said to be* symmetric. *It is also said that e_2 is symmetric to e_1, denoted by $sym(e_1)$.*

Clearly, if e is consistent, then $sym(e)$ is consistent.

The relevant question is whether symmetric expressions have the same normal form; the answer is no.

Proposition 1. *When expressions e_1 and e_2 are symmetric, their normal forms are not always the same.*

We give some examples.

- (Example 1) $\overline{A}ABBAA$ and $AABBA\overline{A}$ are symmetric and have the same normal form $AABB$ (Figure 8).
- (Example 2) $AAAB\overline{B}$ and $\overline{B}BAAA$ are symmetric but have different normal forms: $AABB$ and AAA, respectively (Figure 9).

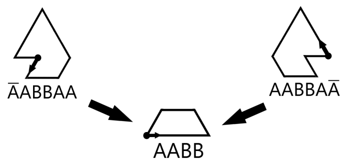


Figure 8: Symmetric expressions with the same normal form.

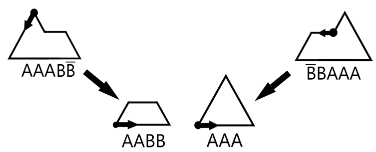


Figure 9: Symmetric expressions with different normal forms.

In Example 2, we recognize two candidate smoothable regions that are to be excised (hatched areas in Figure 10). In fact, we determine such candidates by tracing the outline counter-clockwise. This is why the symmetric expressions may have different normal forms.

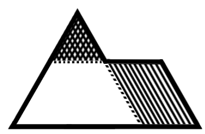


Figure 10: Candidate smoothable regions for excision.

The following proposition holds when symmetric expressions have the same normal form.

Proposition 2. *Let $e = x_1 \ldots x_n$ ($n \geq 3$) be a consistent expression in $\mathcal{D}$, and let e' be its symmetric expression. For each x_i ($i = 1, \ldots, n$), if the following two conditions hold, then e and e' have the same normal form.*

1. *if $x_i = \overline{A}$, then either $x_{i-1}x_ix_{i+1} = A\overline{A}A$ or $x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2} = BB\overline{A}BB$.*

2. *if* $x_i = \overline{B}$*, then* $x_{i-1}x_ix_{i+1} = B\overline{B}B$.

Proof. First, we prove that symmetricity is preserved during each rewriting for all cases.

Let u and v be $x_1 \ldots x_{i-2}$ and $x_{i+2} \ldots x_n$, respectively, and also let w and z be $x_1 \ldots x_{i-3}$ and $x_{i+3} \ldots x_n$, respectively.

Case 1: $x_{i-1}x_ix_{i+1} = A\overline{A}A$

The expression e is $uA\overline{A}Av$, and the expression e' is $sym(v)A\overline{A}Asym(u)$.

The expression e is rewritten to $e_1 = uAv$, and e' is rewritten to $e'_1 = sym(v)Asym(u)$ by applying the same rewriting rule. Note that if $u = u'A\overline{A}$, then $e_1 = u'A\overline{A}Av$; the condition holds after the rewriting. Therefore, e'_1 is a symmetric expression of e_1.

Case 2: $x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2} = BB\overline{A}BB$

The expression e is $wBB\overline{A}BBz$, and the expression e' is $sym(z)BB\overline{A}BBsym(w)$.

The expression e is rewritten to $e_1 = wB\overline{B}\overline{B}Bz$, and e' is rewritten to $e'_1 = sym(z)B\overline{B}\overline{B}Bsym(w)$ by applying the same rewriting rule.

The expression e_1 is rewritten to $e_2 = wBBz$, and e'_1 is rewritten to $e'_2 = sym(z)BBsym(w)$ by applying the same rewriting rule. Therefore, e'_2 is a symmetric expression of e_2.

Case 3: $x_{i-1}x_ix_{i+1} = B\overline{B}B$

As in Case 1, e'_1 is a symmetric expression of e_1.

Therefore, symmetry is preserved during each rewriting.

If we repeatedly apply the same rewriting rules to these symmetric expressions, e and e' are reduced to the normal forms that are symmetric. The symmetric expression of a normal form is itself. Therefore, e and e' have the same normal form. □

5.2 Features of Each Class

Our rewriting system can be regarded as a state transition system in which each state denotes a shape and each transition denotes the generation/deletion of a concavity. From such a viewpoint, our rewriting rule is not intuitive. For example, consider Figure 11. It is natural to think that a figure (a) would result from transformation of a figure (d). However, a figure (a) can be obtained only by starting with (c) and changing through (b). It is interesting to note that similar shapes may be obtained when the initial states differ.

Let e_1 and e_2 be consistent expressions in $\mathcal{D}$. If e_1 and e_2 are reduced to different normal forms, then e_1 is not reduced to e_2 and vice versa. This means that all expressions can be classified into five classes depending on their normal forms. If we can extract features that characterize each class, we can determine the original shape of a given figure.

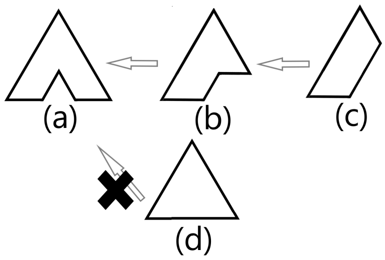


Figure 11: Shape change as a state transition system.

6 CONCLUSION

We proposed a symbolic expression for a qualitative shape of an object in the sequence of rotation angles of edges. We developed a drawing algorithm for the expression based on rewriting strings and proved that, for a consistent expression, it is possible to draw a figure in the two-dimensional plane. The significant point is that we can allocate the vertex so that there is no intersection on the concave part. We refined the algorithm as an abstract rewriting system and proved that the system has both properties of confluence and termination. The contributions of this work are twofold:

- We offer a constructive proof for the existence of a figure corresponding to a symbolic expression.
- We treat spatial data as an abstract rewriting system.

There are several open problems including those mentioned in Section 5. Of these, we are currently considering the following:

- We are exploring other properties of the expressions such as features of a set of expressions that have the same normal form.
- We aim to provide a mechanical proof of confluence and termination using proof assistants such as Coq and Isabelle/HOL.

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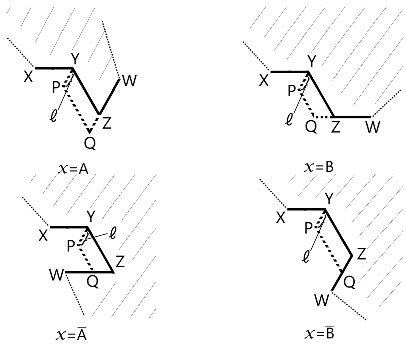
APPENDIX

Proof of Theorem 1. Assume that F is the figure corresponding to e'. Let d be the minimum of the distances between two non-adjacent edges in F. We separate the methods used to smooth e.

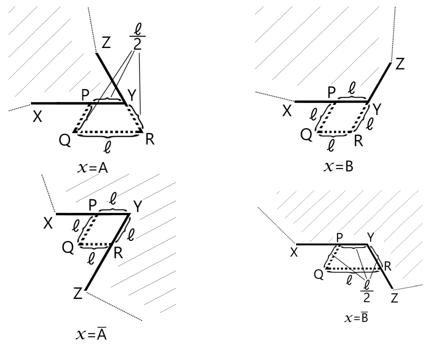
Case 1: When $\overline{B}A$ is replaced by B. Already proven in Section 3.1.

Case 2: When $\overline{A}B$ is replaced by $\overline{B}$. Assume that x is the next rotation angle of $\overline{A}B$ in e. In F, this $\overline{B}x$ is drawn as points X, Y, Z, and W. We can put the points P and Q for each case of x as shown in the following figures.



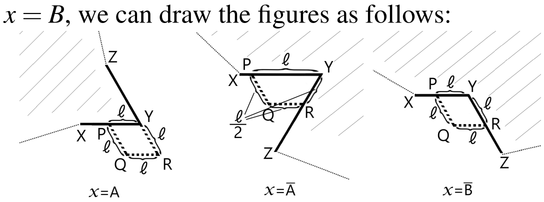
Let l be less than d. Distances between any point on PQ and the edge YZ are less than d; PQ does not intersect with the other edges in F. The region $XYPQW$ corresponds to $\overline{A}Bx$, and the rest part is the same as e'. This means that this figure is denoted by e.

Case 3: When $\overline{A}A$ is replaced by ε. Assume that x is the next rotation angle of $\overline{A}A$ in e. Let l be less than d, and in the case of $x = B$, l be less than $d/\sqrt{3}$. In each case of x, we can draw the figure in a manner similar to the case of $\overline{B}BB$ (described in Section 3.1).



The region $XPQRZ$ corresponds to $\overline{A}Ax$ in each figure; thus, these figures are denoted by e.

Case 4: When $\overline{B}B$ is replaced by ε. Assume that x is the next rotation angle of $\overline{B}B$ in e. We skip the case $x = B$ since we already proved it in Section 3.1. For the other cases, in a manner similar to the case of $x = B$, we can draw the figures as follows:



The region $XPQRZ$ corresponds to $\overline{B}Bx$ in each figure; thus, these figures are denoted by e. □