Qualitative Formalization of a Curve on a Two-Dimensional Plane

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Abstract

We propose a theoretical framework for qualitative spatial representation and reasoning about curves on a two-dimensional plane. We regard a curve as a sequence of segments, each of which has its own direction and convexity, and give a symbolic expression to it. We propose a reasoning method on this symbolic expression; when only a few segments of a curve are visible, we find missing segments by connecting them to create a global smooth continuous curve. In addition, we discuss whether the shape of the created curve can represent that of a real object; if the curve forms a spiral, such a curve is sometimes not appropriate as a border of an object. We show a method that judges the appropriateness of a curve, by considering the orientations of the segments.

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1 Introduction

Various information technology applications are improving the safety and convenience of daily life; these include disaster prevention methods. A study of geological structures can reveal the construction history of geological strata, allowing prediction of future changes to a landscape. Such studies are important for preventing disasters, and provide essential information for understanding the shape of the stratum. However, it is difficult to observe the entire shape of an exposed stratum, and it is almost impossible to collect data at many points. Therefore, we must determine a global stratum that explains the structure of all exposed data. One challenging problem that is frequently encountered in the field of structural geology is identifying a global fold structure that connected local data in the distant past [7].

Assume that three blocks of data, all of which consist of three layers (A, B and C), are collected at different locations as shown in Figure 1 and that "drag fold" structure is observed in layer B because layers A and C slip in the directions indicated by the arrows. If these three pieces of local strata are continuous, how can we determine the global fold structure that combined all of the data?

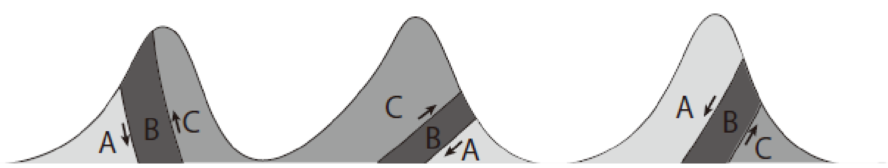

Figure 1 What is a global fold structure?

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Currently, such determinations are performed via numeral simulations that incorporate additional information such as the fossil record and historical climate changes. However, all models are constructed by humans based on their experimental knowledge, which yields varying results because personal interpretations differ. In addition, there is usually insufficient evidence. In this paper, we adopt qualitative spatial reasoning (QSR) as a novel approach to address these concerns. As a result, we derive possible and realistic interpretations using logical reasoning.

QSR is a method of representation and reasoning that focuses on a specific aspect of objects [4, 10, 3, 15]. It is consistent with human recognition, and reduces the computational burden and the amount of required memory because precise numerical values are not employed. This approach is advantageous when treating ambiguity and understanding an abstract state focusing specific aspects of objects such as the relative positional relationships between objects, relative orientation of object movement, abstract shape of an object, and so on. Although many QSR systems have been proposed, few focus on shape, because it is difficult to grasp a shape qualitatively compared to other features. And there have been very few studies on connecting qualitative objects that do not share points or regions. The main issue addressed in this paper is how a curve on a two-dimensional plane can be formalized in a qualitative manner.

In typical image processing, image data are handled by dividing the data into unit cells of the same fixed size. This process aligns multiple consecutive, identical unit data, which leads to redundancy when the goal is the construction of an abstract shape, not a precise shape. A qualitative approach can eliminate this redundancy. Generally, when humans recognize a curve at a glance, the entire shape is understood by dividing the curve into inflection points and extremum points, according to the number of segments.

In this paper, we regard each local data block as a line segment and create a global curve by connecting these line segments. We show formalization that handles both shape and direction when associating all these segments with other segments.

More specifically, the goals of this paper is the formalization of the following two issues:

1. Reasoning about the connection between two line segments and the relative direction between the initial point and terminal points of the derived smooth global curve.
2. Reasoning about the connection between two line segments located in the specific relative direction and the suitable line segments to be inserted to create a smooth global curve.

We also discuss whether a created curve is realistic. If the created curve lies on a two-dimensional plane, does its shape reasonably reflect the border of an actual entity? If the curve is in the form of a logarithmic spiral, it may sometimes not adequately reflect the shape of an object. We show a method that judges the appropriateness of a curve, by considering the orientations of the segments.

The remainder of this paper is organized as follows. In Section 2, we describe the relevant fundamental concepts. In Section 3, we give the rules for direct/indirect segment connections. In Section 4, we describe the connections among segments in specific relative directions. In Section 5, we discuss whether the derived curves are realistic. In Section 6, we apply our method to predict the shape of a global fold stratum. In Section 7, we compare our works with the related studies. Finally, in Section 8, we show the conclusions and describe our future works.

2 Fundamental Concepts

Let *CURVES* be a set of directed curved segments with a unique direction and curvature on a two-dimensional plane.

Let $S_v = \{n, s\}$, $S_h = \{e, w\}$, $Conv = \{cx, cc\}$ and $Dir = S_v \cup S_h$. The symbols n, s, e and w indicate the north, south, east and west directions, respectively, and cx and cc indicate convex and concave, respectively. The direction exactly in the middle between north and south (east and west) is regarded as either n or s (e or w, resp.) Straight lines are not considered.

▶ **Definition 1** (complementary element). *We define a function i that assigns the complementary element in Dir and $Conv$.*

For $Dir \cup Conv$:

- $i(n) = s,\ i(s) = n$
- $i(e) = w,\ i(w) = e$
- $i(cx) = cc,\ i(cc) = cx$

Thus, for each $E \in Dir \cup Conv$, $i(i(E)) = E$.

For $\boldsymbol{X} \in CURVES$, we represent the qualitative shape of $\boldsymbol{X}$ focusing on its intrinsic direction and convexity, ignoring the precise size and the exact curvature.

▶ **Definition 2** (qualitative representation). *For a segment $\boldsymbol{X} \in CURVES$, $X = (V, H, C)$ is said to be* the qualitative representation of $\boldsymbol{X}$ *where $V \in S_v, H \in S_h$ and $C \in Conv$. V, H and C show the vertical direction, horizontal direction and the convexity of $\boldsymbol{X}$, and denoted by $dv(X), dh(X)$ and $cv(X)$, respectively.*

For $\boldsymbol{X}, \boldsymbol{Y} \in CURVES$, let $X = (V, H, C)$ and $Y = (V', H', C')$ be qualitative representations of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. We define the relation $\sim$ on $CURVES$ as follows: $\boldsymbol{X} \sim \boldsymbol{Y}$ iff $V = V'$, $H = H'$ and $C = C'$. Then $\sim$ is an equivalence relation on $CURVES$. As a result, $CURVES$ is classified into eight equivalence classes which are jointly exhaustive and pairwise disjoint. We denote the set of these eight classes as $\mathcal{S}$, that is, $\mathcal{S} = CURVES/\sim$.

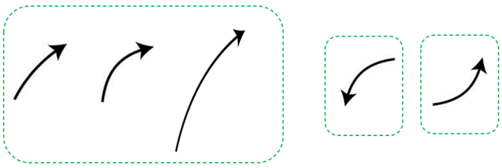


Figure 2 Classes of curved segments.

▶ **Example 3.** In Figure 2, the left three segments are regarded as equivalent, whereas they are different from the right two segments. The qualitative representation of a segment in the leftmost class is (n, e, cx).

3 Connection Rules

3.1 Direct connection

First, we consider the connections between two segments in $\mathcal{S}$ ignoring their relative direction.

For $X \in \mathcal{S}$, its initial and terminal points are indicated by $init(X)$ and $term(X)$, respectively. Note that $init(X)$ and $term(X)$ do not represent the location on a two-dimensional plane, but indicate the parts of X itself. $dir(X) = (dv(X), dh(X))$ indicates the relative direction of $term(X)$ with respect to $init(X)$. For $X, Y \in \mathcal{S}$, $dir(X) = dir(Y)$ iff $dv(X) = dv(Y)$ and $dh(X) = dh(Y)$; $X = Y$, denoted by $ssp(X, Y)$[1] , iff $dir(X) = dir(Y)$ and $cv(X) = cv(Y)$.

[1] *ssp* is an abbreviation of "same shape".

In this paper, a finite continuous smooth (cusp-free) curve without a self-intersection is called an *scurve*. We connect multiple segments to create an scurve.

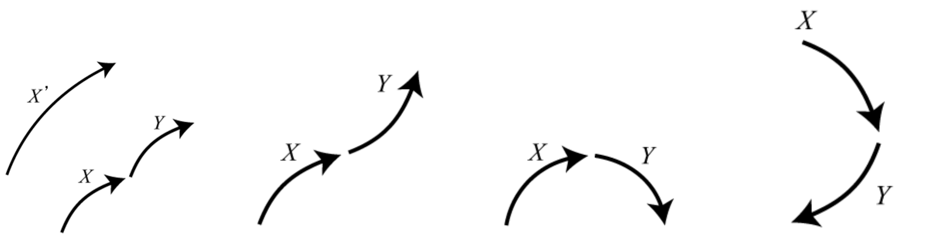
▶ **Definition 4** (directly connectable). *For $X, Y \in \mathcal{S}$, if an scurve is obtained by considering that $init(Y)$ and $term(X)$ are identical, then X and Y are said to be* directly connectable*, and the outcome of the connection is represented as $X \cdot Y$.*

▶ **Example 5.** $X = (n, e, cx)$ and $Y = (s, e, cx)$ are directly connectable, whereas $X = (n, e, cx)$ and $Y = (s, e, cc)$ are not since a cusp is created at their connection.

For $X, Y \in \mathcal{S}$, if $ssp(X, Y)$ holds, X and Y are directly connectable and the result is regarded as a single segment without a cusp, since the precise curvatures of X and Y are ignored. Therefore, it is considered to be $ssp(X \cdot Y, X')$ where $ssp(X, X')$. When X and Y are directly connectable, and if $ssp(X, Y)$ does not hold, the pairs of segments create inflection or extremum points via direct connections. For $X, Y \in \mathcal{S}$, $ifl(X, Y)$ represents that $X \cdot Y$ creates an inflection point (*ifl*), and $xtr(X, Y, D)$ represents that $X \cdot Y$ creates an extremum point (*xtr*) in the direction D, where $D \in Dir$ (Figure 3). These relations are defined as follows.

▶ **Definition 6** (connecting point).

- *$ifl(X, Y)$ iff $X = (V, H, C)$ and $Y = (V, H, i(C))$.*
- *$xtr(X, Y, V)$ iff $X = (V, H, C)$ and $Y = (i(V), H, C)$.*
- *$xtr(X, Y, H)$ iff $X = (V, H, C)$ and $Y = (V, i(H), i(C))$.*



(a) *ssp*. **(b)** *ifl*. **(c)** *xtr* (vertical). **(d)** *xtr* (horizontal).

Figure 3 Directly connectable segments.

$dir(X \cdot Y)$ is represented as a pair (V, H) where $V \in S_v$, $H \in S_h$. $dir(X \cdot Y)$ indicates the relative direction of $term(Y)$ with respect to $init(X)$.

For $X, Y \in \mathcal{S}$, the following properties hold for $X \cdot Y$.

- $ssp(X, Y) \Rightarrow dir(X \cdot Y) = dir(X) = dir(Y)$
- $ifl(X, Y) \Rightarrow dir(X \cdot Y) = dir(X) = dir(Y)$
- $xtr(X, Y, V) \land V \in S_v \Rightarrow dir(X \cdot Y) = (*, dh(X))$
- $xtr(X, Y, H) \land H \in S_h \Rightarrow dir(X \cdot Y) = (dv(X), *)$

Note that the term "$*$" indicates that the value is non-deterministic: for $X, Y \in \mathcal{S}$, when $xtr(X, Y, V)$, where $V \in S_v$ holds, $dv(X \cdot Y) = n$ or s depending on how Y is drawn (Figure 4). Thus, we can draw Y to satisfy $dir(X \cdot Y) = dir(Y)$. This reflects the characteristics of the qualitative treatment.

(a) $dv(X \cdot Y) = n$ **(b)** $dv(X \cdot Y) = s$

Figure 4 Various drawings for $X \cdot Y$.

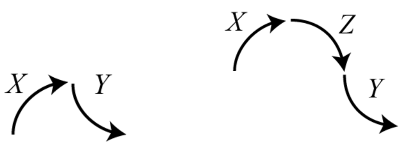
3.2 Indirect connection

When $X, Y \in \mathcal{S}$ are not directly connectable, we insert elements in $\mathcal{S}$ between X and Y to obtain an scurve from X to Y, and extremum points and inflection points are created in the obtained scurve.

▶ **Definition 7** (connectable)**.** *For $X_1, \ldots, X_k \in \mathcal{S}$ $(k \geq 2)$: if for each j such that $1 \leq j \leq k-1$, X_j and X_{j+1} are directly connectable, then X_1 and X_k are said to be* connectable*. The outcome of the connection is an scurve and it is represented as $X_1 \cdot \ldots \cdot X_k$.*

The connection operation "$\cdot$" is associative, that is, $\forall X, Y, Z \in \mathcal{S}; (X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

▶ **Example 8.** $X = (n, e, cx)$ and $Y = (s, e, cc)$ are not directly connectable (Figure 5(a)), but if we insert $Z = (s, e, cx)$ between X and Y, then we get an scurve $X \cdot Z \cdot Y$ (Figure 5(b)).



(a) **(b)**

Figure 5 Indirect connection.

$dir(X_1 \cdot \ldots \cdot X_n)$ is represented as a pair (V, H) where $V \in S_v, H \in S_h$. $dir(X_1 \cdot \ldots \cdot X_n)$ indicates the relative direction of $term(X_n)$ with respect to $init(X_1)$.

▶ **Definition 9** (interpolation number, interpolation segment, shortest scurve, MIN)**.** *An scurve $X_0 \cdot X_1 \cdot \ldots \cdot X_{k+1}$ $(k \geq 1)$ where $X = X_0, Y = X_{k+1}$, is represented as $X - Y$, k is said to be* an interpolation number *of $X - Y$, and $X_1, \ldots, X_k$ are the* interpolation segments *of $X - Y$. An scurve $X - Y$ of which k is the minimum value is said to be* a shortest scurve of $X - Y$ *and k is* the minimum interpolation number (MIN) of $X - Y$*. When X and Y are directly connectable, $X \cdot Y$ is an scurve for which the interpolation number is zero.*

▶ **Example 10.** Assume that $X = (n, e, cx)$ and $Y = (n, w, cc)$. If we take $X_1 = (n, e, cc)$ and $X_2 = (n, w, cx)$, respectively, then $X \cdot X_1 \cdot X_2 \cdot Y$ is an scurve $X - Y$, in which the connections *ifl*, *xtr* and *ifl* appear in this order (Figure 6(a)). If we take $X'_1 = (s, e, cx)$ and $X'_2 = (s, w, cc)$, respectively, then $X \cdot X'_1 \cdot X'_2 \cdot Y$ is also an scurve $X - Y$ in which the connections *xtr*, *xtr* and *xtr* appear in this order (Figure 6(b)). Both are shortest scurves. If we take $X_3 = (s, e, cc)$, then $X \cdot X'_1 \cdot X_3 \cdot X_1 \cdot X_2 \cdot Y$ is also an scurve $X - Y$ in which the connections *xtr*, *ifl*, *xtr*, *xtr* and *ifl* appear in this order (Figure 6(c)).

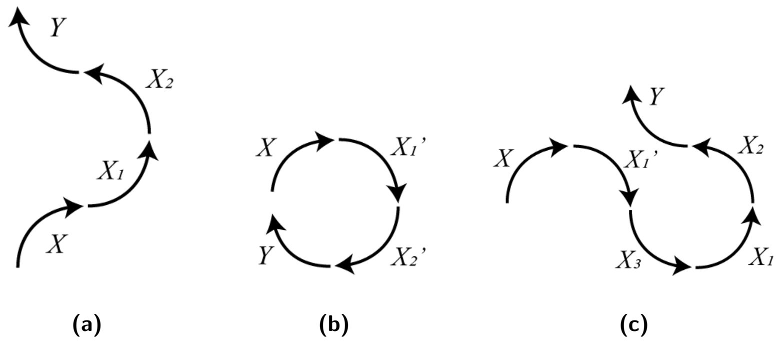


Figure 6 Various scurves $X - Y$.

As the above example illustrates, we can consider infinite kinds of interpolation segments of $X - Y$ for just making an scurve ignoring relative direction of X and Y. The following properties hold between these scurves. For example, the first item states that if two segments are connectable by inserting one interpolation segment and the both connections create *ifl*, then there exists another scurve that directly connects these two segments.

▶ **Proposition 11.**

1. $\forall X \forall Y \forall Z.\ (\ \mathit{ifl}(X, Z) \wedge \mathit{ifl}(Z, Y) \Rightarrow \mathit{ssp}(X, Y)\ )$
2. $\forall X \forall Y \forall Z \forall D.\ (\ \mathit{xtr}(X, Z, D) \wedge \mathit{ssp}(Z, Y) \Rightarrow \mathit{xtr}(X, Y, D)\ )$
3. $\forall X \forall Y \forall Z \forall D.\ (\ \mathit{ssp}(X, Z) \wedge \mathit{xtr}(Z, Y, D) \Rightarrow \mathit{xtr}(X, Y, D)\ )$
4. $\forall X \forall Y \forall Z.\ (\ \mathit{ifl}(X, Z) \wedge \mathit{ssp}(Z, Y) \Rightarrow \mathit{ifl}(X, Y)\ )$
5. $\forall X \forall Y \forall Z.\ (\ \mathit{ssp}(X, Z) \wedge \mathit{ifl}(Z, Y) \Rightarrow \mathit{ifl}(X, Y)\ )$
6. $\forall X \forall Y \forall Z \forall D_1 \forall D_2.\ (\ \mathit{xtr}(X, Z_1, D_1) \wedge \mathit{ifl}(Z_1, Z_2) \wedge \mathit{xtr}(Z_2, Y, D_2) \Rightarrow \mathit{ifl}(X, Y)\ )$
7. $\forall X \forall Y \forall Z_1 \forall Z_2 \forall D_1 \forall D_2 \forall D_3.\ (\ \mathit{xtr}(X, Z_1, D_1) \wedge \mathit{xtr}(Z_1, Z_2, D_2) \wedge \mathit{xtr}(Z_2, Y, D_3) \Rightarrow$
   $\exists Z_3 \exists Z_4 \exists D_4.\ (\ \mathit{ifl}(X, Z_3) \wedge \mathit{xtr}(Z_3, Z_4, D_4) \wedge \mathit{ifl}(Z_4, Y)\ )\ )$
8. $\forall X \forall Y \forall Z_1 \forall Z_2 \forall D_1 \forall D_2.\ (\ \mathit{ifl}(X, Z_1) \wedge \mathit{xtr}(Z_1, Z_2, D_1) \wedge \mathit{xtr}(Z_2, Y, D_2) \Rightarrow$
   $\exists Z_3 \exists Z_4 \exists D_3 \exists D_4.\ (\ \mathit{xtr}(X, Z_3, D_3) \wedge \mathit{xtr}(Z_3, Z_4, D_4) \wedge \mathit{ifl}(Z_4, Y)\ )\ )$

Proof.

1. If $\mathit{ifl}(X, Z)$ and $\mathit{ifl}(Z, Y)$, then $\mathit{dir}(Z) = \mathit{dir}(Y) = \mathit{dir}(X)$, and $\mathit{cv}(X) = i(\mathit{cv}(Y)) = i(i(\mathit{cv}(X)) = \mathit{cv}(X)$. Therefore, $\mathit{ssp}(X, Y)$.

2-5. trivial.

6. Let $X = (V, H, C)$.
   If $D_1 \in S_v$, then $Z_1 = (i(V), H, C)$ from $\mathit{xtr}(X, Z_1, D_1)$. Additionally, $Z_2 = (i(V), H, i(C))$ from $\mathit{ifl}(Z_1, Z_2)$. On the other hand, $D_2 = i(D_1)$ holds, since Z_2 and Y are directly connectable and $\mathit{xtr}(Z_2, Y, D_2)$. Therefore, $Y = (i(i(V)), H, i(C)) = (V, H, i(C))$. Therefore, $\mathit{ifl}(X, Y)$ (Figure 7(a)).
   If $D_1 \in S_h$, then $Z_1 = (V, i(H), i(C))$ from $\mathit{xtr}(X, Z_1, D_1)$. This case is proven, as is the case when $D_1 \in S_v$.
7. Let $X = (V, H, C)$.
   If $D_1 \in S_v$, then $D_2 \in S_h$, $D_3 \in S_v$ and $D_4 \in S_h$. $Z_1 = (i(V), H, C)$ from $\mathit{xtr}(X, Z_1, D_1)$. Then, $Z_2 = (i(V), i(H), i(C))$ from $\mathit{xtr}(Z_1, Z_2, D_2)$. Then, $Y = (V, i(H), i(C))$ from $\mathit{xtr}(Z_2, Y, D_3)$ (Figure 7(b)). On the other hand, $Z_3 = (V, H, i(C))$ from $\mathit{ifl}(X, Z_3)$. Then, $Z_4 = (V, i(H), C)$ from $\mathit{xtr}(Z_3, Z_4, D_4)$. Then, $Y = (V, i(H), i(C))$ from $\mathit{ifl}(Z_4, Y)$ (Figure 7(c)).

If $D_1 \in S_h$, then $D_2 \in S_v$, $D_3 \in S_h$ and $D_4 \in S_v$. $Z_1 = (V, i(H), i(C))$ from $xtr(X, Z_1, D_1)$. Then, $Z_2 = (i(V), i(H), i(C))$ from $xtr(Z_1, Z_2, D_2)$. Then, $Y = (i(V), H, C)$ from $xtr(Z_2, Y, D_3)$ (Figure 7(d)). On the other hand, $Z_3 = (V, H, i(C))$ from $ifl(X, Z_3)$. Then, $Z_4 = (i(V), H, i(C))$ from $xtr(Z_3, Z_4, D_4)$. Then, $Y = (i(V), H, C)$ from $ifl(Z_4, Y)$ (Figure 7(e)).

8. This is proven in a similar manner to the proof shown for case 7.

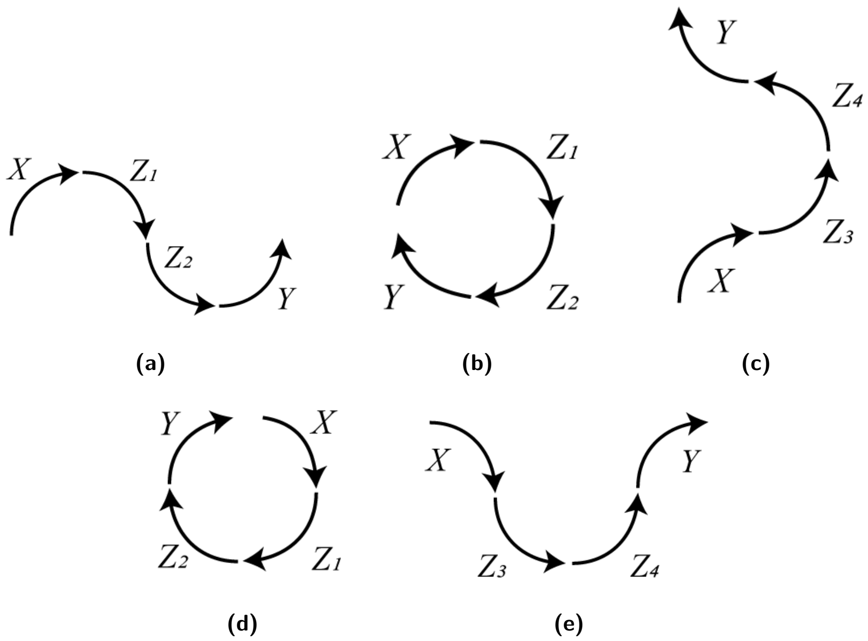


Figure 7 Scurves created by indirect connection.

◀

If an scurve $X - Y$ is obtained of which the interpolation number is more than two, then there exists another scurve of which the interpolation number is less than three. Moreover, there exist more than one scurve $X - Y$ with the same interpolation number but consisting of different interpolation segments. The following can be proven from Proposition 11 and the associativity of the "$\cdot$" operation.

▶ **Proposition 12.** *For $X, Y \in \mathcal{S}$, MIN of $X - Y$ is less than three and it is determined by the folllowing rules.*

- *MIN of $X - Y$ is zero iff $ssp(X, Y)$, $ifl(X, Y)$ or $xtr(X, Y, D)$ holds.*
- *MIN of $X - Y$ is one iff either of the followings holds.*
  - *$\exists Z \exists D.\ xtr(X, Z, D)$ and $ifl(Z, Y)$ (Figure 8(a))*
  - *$\exists Z \exists D_1 \exists D_2.\ xtr(X, Z, D_1)$ and $xtr(Z, Y, D_2)$ (Figure 8(b))*
  - *$\exists Z \exists D.\ ifl(X, Z)$ and $xtr(Z, Y, D)$ (Figure 8(c))*
- *MIN of $X - Y$ is two iff either of the followings holds.*
  - *$\exists Z_1 \exists Z_2 \exists D.\ ifl(X, Z_1)$ and $xtr(Z_1, Z_2, D)$ and $ifl(Z_2, Y)$ (Figure 8(d))*
  - *$\exists Z_1 \exists Z_2 \exists D_1 \exists D_2.\ xtr(X, Z_1, D_1)$ and $xtr(Z_1, Z_2, D_2)$ and $ifl(Z_2, Y)$ (Figure 8(e))*

Proof. For each segment, there are two possible segments that are directly connectable except for the one in *ssp* relation, which create an inflection point (*ifl*) and an extreme point (*xtr*), respectively. Therefore, considering an scurve that consists of four segments, and that

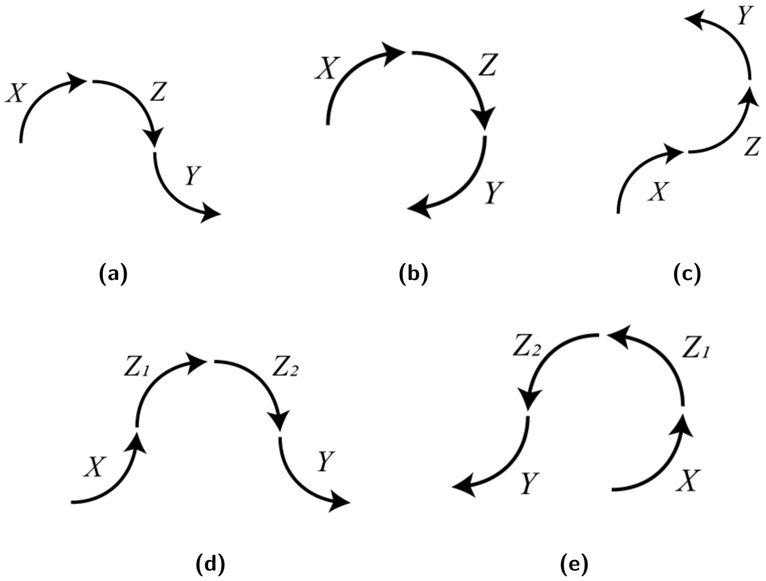


Figure 8 Connection patterns.

starts from an arbitrary shape X, there are $2^3 = 8$ possible scurves (Figure 9). All the eight elements in $\mathcal{S}$ can be found as a segment of either of these scurves. It means that each $Y \in \mathcal{S}$ is connectable from X, and MIN of $X - Y$ is less than three. Therefore, we investigate scurves consisting of less than five segments.

When an scurve $X - Y$ consists of three segments, it has two connection points. If both of them are *ifl*, then there exists another scurve $X - Y$ for which $ssp(X, Y)$ holds, from the first item of Proposition 11. Therefore, MIN is zero in this case. If one of the connection points is *xtr*, there does not exist other scurve of which the interpolation number is less than one; therefore, MIN is one in these cases.

When an scurve $X - Y$ consists of four segments, it has three connection points, each of which is either *ifl* or *xtr*.

- If *ifl*, *ifl*, *ifl* appear in this order, then there exists another scurve $X - Y$ for which $\mathit{ifl}(X, Y)$ holds, from the first and fifth items of Proposition 11. MIN is zero in this case.
- If *ifl*, *ifl*, *xtr* appear in this order, then there exists another scurve $X - Y$ for which $xtr(X, Y, D)$ holds, from the first and third items. MIN is zero in this case.
- If *ifl*, *xtr*, *xtr* appear in this order, then there exists another scurve $X - Y$ in which *xtr*, *xtr*, *ifl* appear in this order, from the eighth item.
- If *xtr*, *ifl*, *ifl* appear in this order, then there exists another scurve $X - Y$ for which $xtr(X, Y, D)$ holds, from the first and second items. MIN is zero in this case.
- If *xtr*, *ifl*, *xtr* appear in this order, then there exists another scurve $X - Y$ for which $\mathit{ifl}(X, Y)$ holds, from the sixth item. MIN is zero in this case.
- If *xtr*, *xtr*, *xtr* appear in this order, then there exists another scurve $X - Y$ in which *ifl*, *xtr*, *ifl* appear in this order, from the seventh item.

Thus, the scurves consisting of four segments that cannot be reduced to directly connectable scurves are reduced to one of the scurves in which *ifl*, *xtr*, *ifl* or *xtr*, *xtr*, *ifl* appear in these orders. For these two cases, there does not exist other scurve of which the interpolation number is less than two; therefore, MIN is two in these cases.

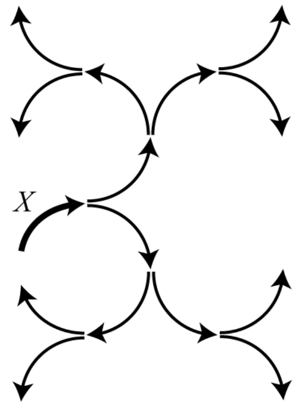


Figure 9 Eight kinds of scurves consisting of four segments.

◀

Therefore, the following proposition holds.

▶ **Proposition 13.** *For any $X, Y \in \mathcal{S}$, MIN of $X - Y$ is either zero, one or two.*

Proof. It is derived from Proposition 12 straightforwardly. ◀

4 Designated Relative Direction

We have investigated the shortest scurve that indirectly connects $X, Y \in \mathcal{S}$, and discussed the relative direction of X and Y in the obtained scurve. Here, we discuss the interpolated segments and the interpolation number of $X - Y$, when the relative direction of X and Y is given.

For $X, Y \in \mathcal{S}$, we introduce the relation $rdir(X, Y)$ that is the relative direction of Y with respect to X.

$rdir(X, Y)$ is represented as a pair (V, H), where $V \in S_v$, $H \in S_h$. $rdir(X, Y)$ indicates the relative direction of $init(Y)$ with respect to $term(X)$.

Let $rdir(X_1, Y_1) = (V_1, H_1), rdir(X_2, Y_2) = (V_2, H_2)$ and $dir(X) = (V, H)$. Then, $rdir(X_1, Y_1) = rdir(X_2, Y_2)$ iff $V_1 = V_2$ and $H_1 = H_2$; and $rdir(X_1, Y_1) = dir(X)$ iff $V_1 = V$ and $H_1 = H$.

▶ **Example 14.** If $ssp(X, Y)$, then X and Y are directly connectable; however, if the condition $rdir(X, Y) = (s, w)$ is added, X and Y cannot be directly connected in this direction (Figure 10).

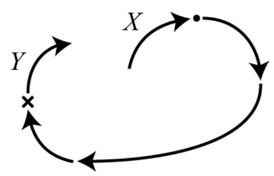


Figure 10 Connection of segments under the designated relative direction.

In a case such as Example 14, we must insert elements between X and Y to satisfy the designated direction $rdir(X,Y)$.

▶ **Definition 15** (interpolation number on $rdir$, shortest scurve on $rdir$, RMIN). *For an scurve $X_0 \cdot X_1 \cdot \ldots \cdot X_{k+1}$ ($k \geq 1$), where $X = X_0, Y = X_{k+1}$, that satisfies a given $rdir(X,Y)$, k is said to be* an interpolation number on $rdir$ of $X - Y$. *An scurve $X - Y$ that satisfies $rdir(X,Y)$ of which k is the minimum is said to be* a shortest scurve of $X - Y$ on $rdir$, *and k is said to be* the minimum interpolation number on $rdir$ (RMIN) of $X - Y$.

Here, we discuss RMIN of $X - Y$ when $rdir(X,Y)$ is given, by the following procedure. First, for $X, Y \in \mathcal{S}$, we insert segments X' and Y' that satisfy $dir(X') = dir(Y') = rdir(X', Y') = rdir(X,Y)$ between X and Y, and consider the connection by dividing the segments into three parts, $X - X', X' - Y'$ and $Y' - Y$; and then combine them.

In the following, we discuss the RMINs of these three parts, respectively.

[1] $X' - Y'$

There exist two distinct shapes X' of which the convexities differ, and there exist two distinct shapes Y' of which the convexities differ. For each combination of these shapes, RMIN of $X' - Y'$ is zero.

[2] $X - X'$

▶ **Proposition 16.** *Let k be RMIN of $X - X'$. Then $0 \leq k \leq 1$ holds.*

Proof. We will show that the proposition holds by dividing the case depending on the shape of X. We can take X' that satisfies either $cv(X') = cx$ or cc. Figure 11 illustrates the case in which $dir(X') = (s, e)$. In the first two cases, the direction of $term(X')$ with respect to $term(X)$ is always $dir(X')$, while in the other six cases, we can draw X' so that the direction of $term(X')$ with respect to $term(X)$ is $dir(X')$. Let $X' = (V, H, C)$.

1. $X = (V, H, cx)$.
   Take X' such that $cv(X') = cx$. X and X' are directly connectable; $k = 0$ (Figure 11(a)).
2. $X = (V, H, cc)$.
   Take X' such that $cv(X') = cc$. X and X' are directly connectable; $k = 0$ (Figure 11(b)).
3. $X = (i(V), H, cx)$.
   Take X' such that $cv(X') = cx$. As $xtr(X, X', i(V))$ holds, X and X' are directly connectable; $k = 0$ (Figure 11(c)).
4. $X = (i(V), H, cc)$.
   Take X' such that $cv(X') = cx$. If we take $Z = (i(V), H, C)$, then $ifl(X, Z)$ and $xtr(Z, X', i(V))$ hold; $k = 1$ (Figure 11(d)).
5. $X = (i(V), i(H), cx)$.
   Take X' such that $cv(X') = cc$. If we take $Z = (V, i(H), cx)$, then $xtr(X, Z, i(V))$ and $xtr(Z, X', i(H))$ hold; $k = 1$ (Figure 11(e)).
6. $X = (i(V), i(H), cc)$.
   Take X' such that $cv(X') = cx$. If we take $Z = (V, i(H), cx)$, then $xtr(X, Z, i(H))$ and $xtr(Z, X', i(V))$ hold; $k = 1$ (Figure 11(f)).
7. $X = (V, i(H), cx)$.
   Take X' such that $cv(X') = cc$. As $xtr(X, X', i(H))$ holds, X and X' are directly connectable; $k = 0$ (Figure 11(g)).
8. $X = (V, i(H), cc)$.
   Take X' such that $cv(X') = cc$. If we take $Z = (V, i(H), cx)$, then $ifl(X, Z)$ and $xtr(Z, X', i(H))$ hold; $k = 1$ (Figure 11(h)).

Therefore, the proposition holds. ◀

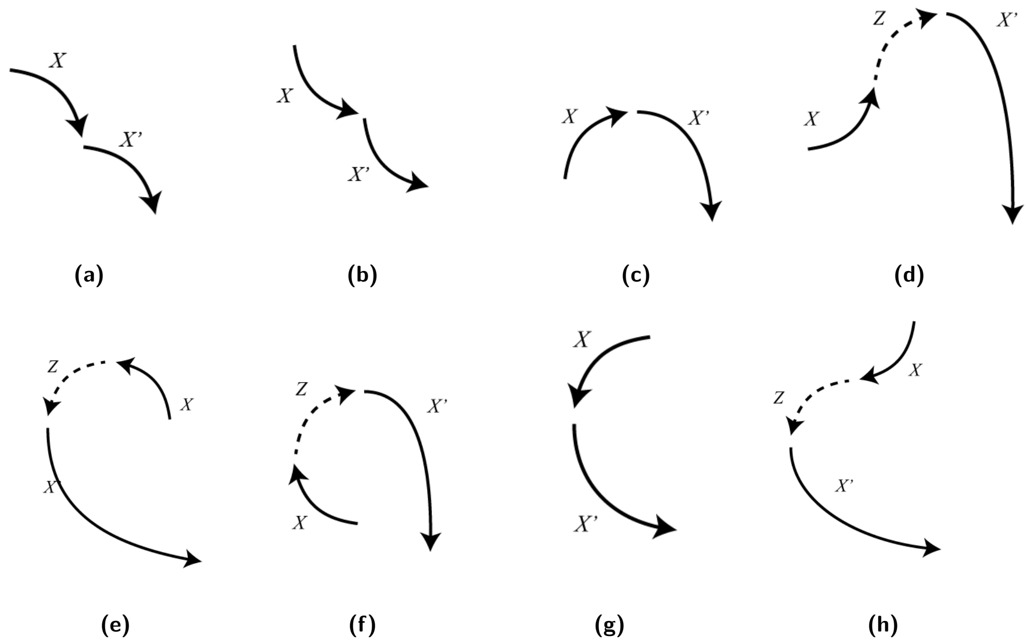


Figure 11 Connection $X - X'$ for $rdir(X,Y) = (s,e)$.

[3] $Y' - Y$

Similarly, we can draw Y' so that the direction of $init(Y)$ with respect to $init(Y')$ is $dir(Y')$, and RMIN of $Y' - Y$ is less than two.

Finally, we combine the results of [1]-[3].

The number of segments that make up $X-Y$ ranges from four to six by adding X', Y', X and Y, from the above discussion. Let $X_1 \cdot \ldots \cdot X_n$ be a shortest scurve $X - Y$ on $rdir$ that is created by the above procedure, where $X = X_1$ and $Y = X_n$. When $ssp(X_{i-1}, X_i)$ $(\forall i; 1 < i \leq n)$, X_{i-1} and X_i can be merged into one; and it may occur in the connection with X', Y', X and Y.

Finally, the number of segments that make up $X - Y$ ranges from one to six.

▶ **Theorem 17.** *For $X, Y \in \mathcal{S}$, when $rdir(X,Y)$ is given, the number of segments that configure a shortest scurve of $X - Y$ ranges from one to six.*

5 Intersection of a Curve

We have shown that an scurve can be obtained that satisfies the designated relative direction by connecting any pair of line segments. However, is this scurve a natural curve that might represent the border of an entity in the real world? If the scurve is in the form of a logarithmic spiral, then the scurve is not an appropriate border for an actual object.

Assume that for $X, Y \in \mathcal{S}$, $rdir(X,Y)$ is given. There exist an infinite number of drawings of X and Y on a two-dimensional plane when the qualitative approach is taken. Here, we discuss the existence of a drawing such that the obtained curve does not form a spiral on a two-dimensional plane.

5.1 Admissibility of an scurve

In the following, "draw X" means an assignment of one $\boldsymbol{X} \in CURVES$ to $X \in \mathcal{S}$.

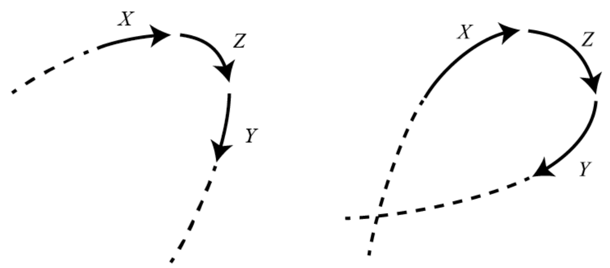
▶ **Definition 18** (drawing).
1. *Let $\boldsymbol{X} \in CURVES$ be a curved segment on a two-dimensional plane of which $X \in \mathcal{S}$ is its qualitative representation. (Note that there are infinitely many $\boldsymbol{X}$'s.) Then $\boldsymbol{X}$ is said to be* a drawing of X. *$init(\boldsymbol{X})$ and $term(\boldsymbol{X})$ represent the locations of the initial point and the terminal point of $\boldsymbol{X}$ on a two-dimensional plane, respectively.*
2. *Let $X_1 \cdot \ldots \cdot X_n$ be an scurve $X_1 - X_n$, and $\boldsymbol{X}_i$ $(1 \leq i \leq n)$ be* a drawing of X_i. *For all i such that $1 \leq i \leq n-1$, if $term(\boldsymbol{X}_i)$ and $init(\boldsymbol{X}_{i+1})$ are located in the same position, then $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_n$ is said to be* a drawing of the scurve $X_1 - X_n$.

First, we introduce the concepts of open and closed drawings of an scurve.

▶ **Definition 19** (closed, open). *For $X, Y \in \mathcal{S}$, let C be a drawing of an scurve $X - Y$ on a two-dimensional plane, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ be drawings of X and Y, respectively. And C' be an infinite-length curve that is obtained by extending C in both directions in a manner such that the curvatures of $\boldsymbol{X}$ at $init(\boldsymbol{X})$ and $\boldsymbol{Y}$ at $term(\boldsymbol{Y})$ are preserved. If C' has a self-intersection point, then the drawing is said to be* closed*; otherwise, it is* open.

▶ **Definition 20** (admissible). *If there is an open drawing for an scurve, then the scurve is said to be* admissible.

▶ **Example 21.** Figure 12(a) and (b) shows two kinds of drawings of an scurve $X \cdot Z \cdot Y$ that are open and closed, respectively, when $X = (n, e, cx)$, $Z = (s, e, cx)$, $Y = (s, w, cc)$ and $rdir(X, Y) = (s, e)$. Therefore, $X \cdot Z \cdot Y$ is admissible.



(a) open. **(b)** closed.

Figure 12 Two drawings of $X \cdot Z \cdot Y$.

For $X \in \mathcal{S}$, we define clockwise ("$+$") and anticlockwise ("$-$") orientations.

▶ **Definition 22** (orientation). *For $X \in \mathcal{S}$, $orn(X) = +$ iff $X = (n, e, cx)$, (s, e, cx), (s, w, cc) or (n, w, cc); and $orn(X) = -$ iff $X = (s, w, cx)$, (s, e, cc), (n, e, cc) or (n, w, cx). For an scurve $X_1 \cdot \ldots \cdot X_n$, the orientation $orn(X_1 \cdot \ldots \cdot X_n)$ is represented as the sequence $orn(X_1) \ldots orn(X_n)$.*

▶ **Example 23.** In Example 21, $orn(X \cdot Z \cdot Y) = +++$.

Before discussing the relationship between the admissibility of an scurve and its orientation, we describe the basic properties of a sequence of orientations.

For the orientation of an scurve p, let d be the difference between the numbers of "+" and "−" that appear in $orn(p)$. If $d \geq 4$, the scurve is not admissible, since the size of the rotation angle is greater than or equal to 2π.

▶ **Definition 24** (uni-directional). *Let p be an scurve $X_1 \cdot \ldots \cdot X_n$, where $n \geq 3$. If for all i ($1 \leq i \leq n$), $dv(X_i)$ is the same or $dh(X_i)$ is the same, then we say that p is* uni-directional.

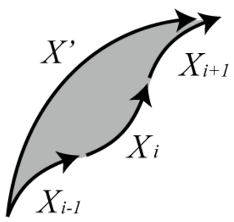
When $orn(p)$ includes the sequence $+-+$, then the open/closed property is preserved if we replace this sequence by $+$; and the same holds for $-+-$.

▶ **Proposition 25.** *Let p be an scurve $X_1 \cdot \ldots \cdot X_n$, where $n \geq 3$.*

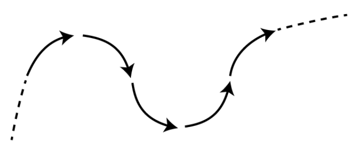
1. *Let $orn(p) = C_1 \ldots C_n$ where $C_{i-1} = C_{i+1} = +$ and $C_i = -$ hold, or $C_{i-1} = C_{i+1} = -$ and $C_i = +$ holds, for some i ($2 \leq i \leq n-1$). Assume that p' is an scurve $X_1 \cdot \ldots \cdot X_{i-2} \cdot X' \cdot X_{i+2} \cdot \ldots \cdot X_n$, such that $ssp(X', X_{i-1})$ holds. Then, p' is admissible iff p is admissible.*
2. *If p is uni-directional, then the scurve is admissible.*

Proof.

1. Assume that p is admissible. As $ifl(X_{i-1}, X_i)$ and $ifl(X_i, X_{i+1})$ hold, $dir(X_{i-1} \cdot X_i \cdot X_{i+1}) = dir(X_{i-1})$. We can take $\boldsymbol{X}'$ by regarding $init(\boldsymbol{X}') = init(\boldsymbol{X}_{i-1})$ and $term(\boldsymbol{X}') = term(\boldsymbol{X}_{i+1})$ (Figure 13(a)). The hatched region in this figure may be made as thinly as required, depending on the drawings of X_{i-1}, X_i and X_{i+1}. Therefore, p' can be drawn without an intersection point. Conversely, assume that p' is admissible. Take a drawing of p' in which the curvature of $\boldsymbol{X}'$ is sufficiently small. Then we can draw X_{i-1}, X_i and X_{i+1} so that their curvatures are sufficiently small. Therefore, p can be drawn without an intersection point. Therefore, p' is admissible iff p is admissible.
2. Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_n$ be drawings of X_1 and X_n, respectively. If we extend the scurve on a two-dimensional plane at the points $init(\boldsymbol{X}_1)$ and $term(\boldsymbol{X}_n)$, respectively, then the scurve is extended in the vertically or horizontally opposite direction. Therefore, the extended parts do not intersect with each other or with $\boldsymbol{X}_1 \cdot \ldots \cdot \boldsymbol{X}_n$ (Figure 13(b)). Therefore, p is admissible.



(a)



(b)

Figure 13 Properties of scurve orientations.

◀

5.2 Judgment of admissibility

For any $X, Y \in \mathcal{S}$ and $rdir(X, Y)$, the number of segments included in a shortest scurve of $X - Y$ ranges from one to six from Theorem 17. Therefore, we investigate scurves consisting of one to six segments. We investigate the minimum possible patterns, considering the clockwise–anticlockwise symmetry and the left–right symmetry.

Let p be an scurve $X_1 \cdot \ldots \cdot X_n$ ($1 \leq n \leq 6$), and d be the difference between the numbers of "+" and "−" that appear in $orn(p)$.

5.2.1 Less than three segments

When $n = 1$ or $n = 2$, p is trivially admissible.

5.2.2 Three segments

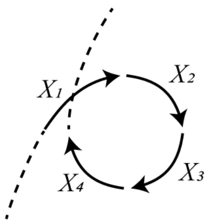
When $orn(X_1 \cdot X_2 \cdot X_3) = +++$, there exists the open drawing shown in Figure 12(a). Therefore, the scurve is admissible.

When $orn(X_1 \cdot X_2 \cdot X_3) = ++-$, the scurve is uni-directional. Therefore, the scurve is admissible, from Proposition 25.

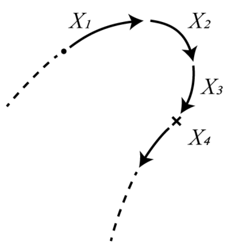
When $orn(X_1 \cdot X_2 \cdot X_3) = +-+$, then the admissibility of this scurve is reduced to that of the scurve consisting of only one segment X_1' such that $orn(X_1') = +$, from Proposition 25. Therefore, the scurve is admissible.

5.2.3 Four segments

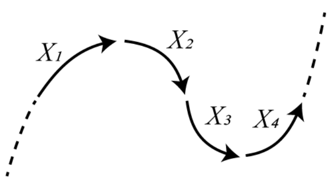
1. When $orn(p) = ++++$, since $d \geq 4$, the scurve is not admissible (Figure 14(a)).
2. When $orn(p) = +++-$, the connections xtr, xtr, and ifl appear in this order in p. There exists an open drawing for the scurve $p' = X_1 \cdot X_2 \cdot X_3$, where $orn(p') = +++$. It follows that we can draw X_3 so that $term(\boldsymbol{X}_3)$ can be located in a direction ensuring that a drawing of p' does not have a self-intersection point. Moreover, as the segments X_3 and X_4 create an inflection point, we can draw X_4 such that a drawing of p does not have a self-intersection point. Therefore, there exists an open drawing for p (Figure 14(b)). Therefore, p is admissible.
3. When $orn(p) = ++--$, the connections xtr, ifl and xtr appear in this order in p. The scurve is uni-directional (Figure 14(c)). Therefore, p is admissible



(a) $++++$



(b) $+++-$



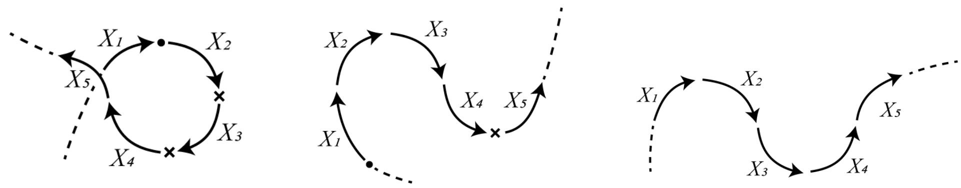
(c) $++--$

Figure 14 Orientations of scurves with four segments.

5.2.4 Five segments

1. When $orn(p) = +++++$, since $d \geq 4$, it is not admissible.
2. When $orn(p) = ++++-$, the connections xtr, xtr, xtr and ifl appear in this order in p. Let $dir(X_1) = (V, H, C)$, $dir(X_2) = (V_2, H_2)$ and $dir(X_2 \cdot X_3) = (V_3, H_3)$. Then $V_2 = V_3 = i(V)$ or $H_2 = H_3 = i(H)$ holds. When $V_2 = V_3 = i(V)$, $X_5 = (V, i(H), C)$ holds; and when $H_2 = H_3 = i(H)$, $X_5 = (i(V), H, C)$ holds. In both cases, any drawing of p has a self-intersection, namely, closed (Figure 15(a)).

3. When $orn(p) = +++--$, the connections xtr, xtr, ifl and xtr appear in this order in p. $X_2 \cdot X_3 \cdot X_4 \cdot X_5$ is uni-directional. Let $dir(X_5) = (V, H)$ and $dir(X_1 \cdot X_2 \cdot X_3 \cdot X_4) = (V_4, H_4)$. Then we can draw X_1 so that $V = i(V_4)$ or $H = i(H_4)$ holds. In both cases, the infinite curve obtained by extending a drawing of p has no self-intersection (Figure 15(b)). Therefore, p is admissible.
4. When $orn(p) = ++--+$, the connections xtr, ifl, xtr and ifl appear in this order in p. p is uni-directional (Figure 15(c)). Therefore, p is admissible.



(a) $++++-$ **(b)** $+++--$ **(c)** $++--+$

Figure 15 Orientations of scurves with five segments.

5. Otherwise, since $+-+$ is always included in $orn(p)$, the admissibility is reduced to that of the scurve of which the length is three, from Proposition 25. Therefore, all the scurves are admissible.

5.2.5 Six segments

1. When $d \geq 4$, the scurve is not admissible.
2. When $orn(p) = ++++--$, the connections xtr, xtr, xtr, ifl and xtr appear in this order in p. There exists an open drawing for the scurve $p' = X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot X_6$, where $orn(p') = +++--$. It follows that we can draw X_2 so that $init(\boldsymbol{X}_2)$ can be located in a direction ensuring that a drawing of p' does not have a self-intersection point. Let $dir(X_1) = (V, H)$ and $dir(p') = (V_2, H_2)$. Then we can draw X_1 such that $V = i(V_2)$ or $H = i(H_2)$ holds. In both cases, the infinite curve obtained by extending a drawing of p has no self-intersection point (Figure 16(a)). Therefore, p is admissible.
3. When $orn(p) = ++--++$, the connections xtr, ifl, xtr, ifl and xtr appear in this order in p. As p is uni-directional, p is admissible (Figure 16(b)).
4. When $orn(p) = +++--+$, the connections xtr, xtr, ifl, xtr and ifl appear in this order in p. There exists an open drawing for the scurve $p' = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5$, where $orn(p') = +++--$. Since the segments X_5 and X_6 create an inflection point, we can draw X_6 such that a drawing of p does not have a self-intersection point (Figure 16(c)). Therefore, p is admissible.
5. When $orn(p) = -++++-$, the connections ifl, xtr, xtr, xtr and ifl appear in this order in p. Let $p' = X_2 \cdot X_3 \cdot X_4 \cdot X_5$. Also let $dir(X_2) = (V, H)$ and $dir(p') = (V', H')$. We can draw p' so that either $V = i(V')$ or $H = i(H')$ holds depending on whether the extremum point of $X_2 \cdot X_3$ is vertical or horizontal. And we can connect $\boldsymbol{X}_1$ and $\boldsymbol{X}_6$ with small convexities to the both ends of this drawing of p', respectively. Then we get a drawing of p that is open, in both cases (Figure 16(d)). Therefore, p is admissible.
6. When $orn(p) = +++---$, the connections xtr, xtr, ifl, xtr and xtr appear in this order in p. There exists an open drawing for the scurve $p' = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5$, where $orn(p') = +++--$. It follows that we can draw X_5 so that $term(\boldsymbol{X}_5)$ can be located in a direction ensuring that a drawing of p' does not have a self-intersection point. Let

$dir(X_2) = (V, H)$ and $dir(X_3 \cdot X_4 \cdot X_5 \cdot X_6) = (V_6, H_6)$. Then, we can draw X_6 such that $V_6 = V$ or $H_6 = H$ holds. In both cases, the infinite curve obtained by extending a drawing of p has no self-intersection point (Figure 16(e)). Therefore, the scurve is admissible.

7. When $orn(p) = ++---+$, the connections *xtr*, *ifl*, *xtr*, *xtr* and *ifl* appear in this order in p. There exists an open drawing for the scurve $p' = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5$, where $orn(p') = ++---$. As the segments X_5 and X_6 create an inflection point, we can draw X_6 such that a drawing of p does not have a self-intersection point (Figure 16(f)). Therefore, p is admissible.

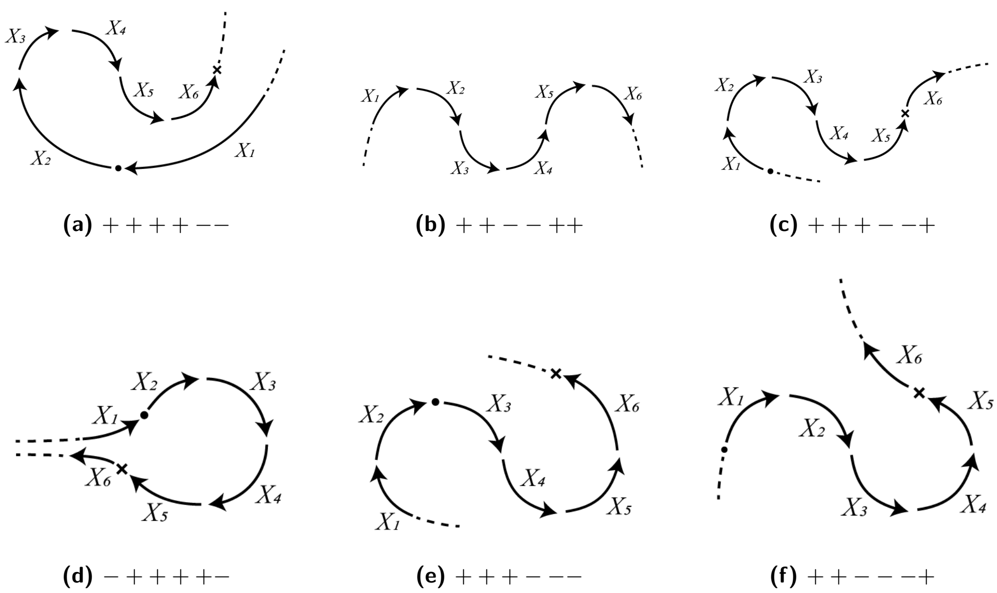


Figure 16 Orientations of scurves with six segments.

8. Otherwise, $+-+$ is always included and more than one $-$ is always included in $orn(p)$. The admissibility of the scurve is reduced to that of an scurve of length four, which includes at least one $-$, from Proposition 25. Therefore, all scurves are admissible.

5.3 Summary

The reasoning on admissibility of scurves is similar if we exchange $+$ and $-$, or reverse the order of the sequence. Thus, the patterns shown in the above subsections cover all possible scurve orientations.

In summary, we have the following property.

▶ **Theorem 26.** *For the scurve p that connects a given X, Y satisfying $rdir(X, Y)$, the following property holds.*

Let d be the difference between the numbers of "$+$" and "$-$" that appear in $orn(p)$.

1. *When $d \geq 4$, then p is not admissible.*
2. *When $orn(p)$ is either $++++-$, $-++++$, $----+$ or $+----$, p is not admissible.*
3. *Otherwise, p is* admissible.

6 Application

We apply our method to predict the shape of the global fold stratum mentioned in Section 1.

Assume that three pieces of data, all of which feature three layers, are collected at different locations (Figure 17(a)). If these three pieces of local strata are continuous, then how can we determine the global fold structure that combines all of these data?

This problem is formalized as follows. For $X, Y, Z \in \mathcal{S}$ where $X = (s, e, cc), Y = (n, e, cc), Z = (s, w, cc)$, $rdir(X, Y) = (n, e)$ and $rdir(Y, Z) = (n, e)$ are given, find an scurve that connects them (Figure 17(b)). Here, the relative directions are determined by the observed drag fold.

Following the reasoning described in Section 4, we obtain the connection $X \cdot X_1 \cdot X_2 \cdot Y \cdot Y_1 \cdot Y_2 \cdot Y_3 \cdot Z$ (Figure 17(c)). This connection is reduced to the scurve of five segments $(s, e, cc) \cdot (n, e, cc) \cdot (n, e, cx) \cdot (s, e, cx) \cdot (s, w, cc)$ by merging consecutive segments in the same shape. The orientation of this scurve is $--+++$, and we find that the scurve is admissible.

As a result, we predict that a global stratum might exist that combines the three pieces of local data, and the abstract shape of the stratum is shown in Figure 17(d).

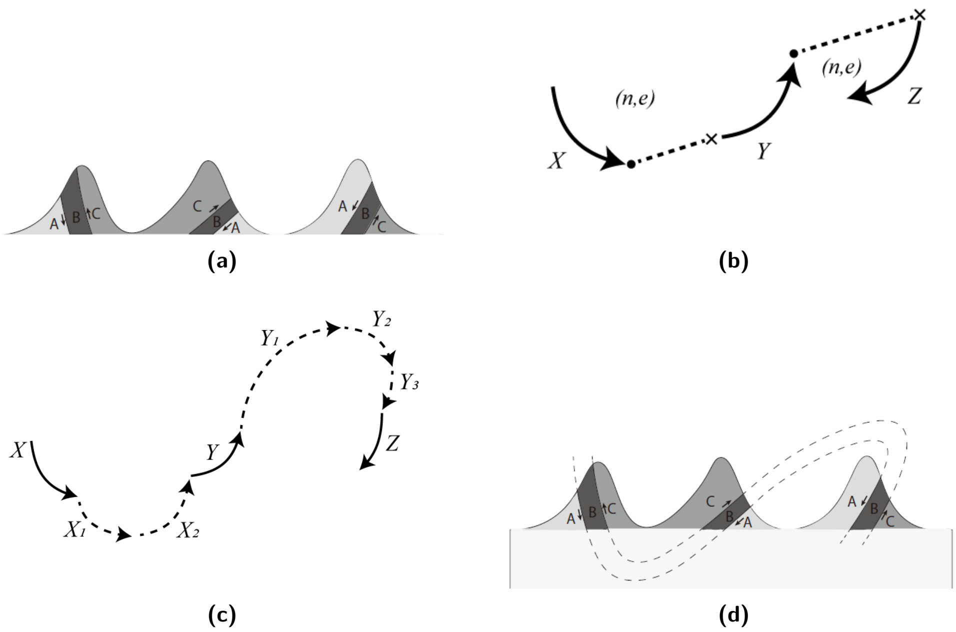


Figure 17 Application of the method to stratum prediction.

7 Related Works

Geological structures are usually investigated employing simulations and logical formalization or reasoning is almost never used. Exceptionally, Shiono et al. [14] proposed a logical model of a geological structure based on stratigraphy and configuration from a geological perspective. Although they provided a mathematical basis for the construction of a geological map, this was not a qualitative approach, and they did not focus on representations of shape or directions. Taniuchi et al. [17] presented a qualitative treatment and reasoning for fold strata. They proposed a qualitative representation of a local stratum consisting of multiple layers, and showed how to derive global data from these local data using connection rules. However, they did not treat the relative direction, or intersections of curves on a two-dimensional plane.

Although a great deal of QSR research has been conducted, few studies have focused on shapes, particularly on curves. Leyton [9] proposed a grammar that represented changes in the shape of a closed curve, starting from a simple smooth curve. They explained the changes in shape that result from a force acting inside or outside the curve. It was shown that a smooth closed curve of any shape could be represented using language based on the proposed grammar. Tosue et al. [18] extended this grammar to handle phenomena such as the creation of a tangent point and division of the curve. Galton et al. [6] proposed another grammar that was applicable not only to a smooth curve, but also to a straight line or a curve with cusps. They showed that objects of various shapes could be symbolically represented by connecting a finite number of primitive segments. Cabedo et al. [2] proposed a representation for the border of an object that included further information such as relative lengths and relative angles. The shape of a region is usually represented by tracing the border of an object, and the juxtaposition of objects has been formalized [1, 13, 5]. However, none of these works treated entities at distant locations.

Kulik et al. [8] applied QSR to landscape silhouettes. They proposed a descriptive language that represented the shape of an open line that was the border of a landscape from the horizontal perspective. Additionally, rules were provided that yielded abstractions of the lines by combining refined line segments. However, neither entities at distant locations nor curved lines were treated.

Several systems have been proposed that focused on direction in QSR. Skiadopoulos et al. [16] presented a cardinal direction calculus using the binary relationships of regional directions. Moratz et al. [11, 12] proposed a calculus termed OPRA, which used the ternary relationships of the directions of entities. In OPRA, a primitive object is a vector with an intrinsic direction. Thus, both the initial and terminal points of entities are considered. In this sense, our formalization and OPRA are somewhat similar. However, a primitive object in OPRA does not have shape as an attribute.

8 Conclusion

We have formalized a qualitative treatment of curves and their relative directions, and proposed a system that handles spatial data on a symbolic representation. As a result, we can find missing segments that connect segments separated by distinct distances while respecting the constraints imposed by their relative locations. We have also shown the judgment whether the obtained global curve is realistic or not.

Curves are common in many natural objects, ranging from the micro level (such as cells) to the macro level (such as terrains). We frequently encounter situations in which we need to predict the abstract entire shape of a curve that is only partly disclosed or that includes unclear parts. The method proposed in this study may contribute to the analysis of medical images that are ambiguous or that lack some regions. The condition of the scurve is also required in wiring problems of circuits.

In future, we will implement this reasoning system and apply to issues including the prediction of fold strata. It will be also interesting to consider a qualitative calculus based on the relative directional relations of the line segments of a curve, and to investigate the emerging logical properties.

References

1 Lledó Museros Cabedo, Luis González Abril, Francisco Velasco Morente, and Zoe Falomir. A pragmatic qualitative approach for juxtaposing shapes. *Journal of Universal Computer Science*, 16(11):1410–1424, 2010.

2 Lledó Museros Cabedo and M. Teresa Escrig. A qualitative theory for shape representation and matching for design. In *Proceedings of the Sixteenth Eureopean Conference on Artificial Intelligence, ECAI'2004*, pages 858–862, 2004.

3 Juan Chen, Anthony G. Cohn, Dayou Liu, Shengsheng Wang, Jihong Ouyang, and Qiangyuan Yu. A survey of qualitative spatial representations. *The Knowledge Engineering Review*, 30:106–136, 2013.

4 Anthony G. Cohn and Jochen Renz. *Handbook of Knowledge Representation*. Elsevier, 2008.

5 Zoe Falomir, Albert Pich, and Vicent Costa. Spatial reasoning about qualitative shape compositions. *Annals of Mathematics and Artificial Intelligence*, 88(11):589–621, 2020.

6 Antony Galton and Richard Meathrel. Qualitative outline theory. *Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence*, pages 1061–1066, 1999.

7 Kenichi Kano and Akihiro Murata. *Structural Geology (in Japanese)*. Asakura Publishing Co., Ltd., 1998.

8 Lars Kulik and Max J. Egenhofer. Linearized terrain: languages for silhouette representations. In *Spatial Information Theory. Foundations of Geographic Information Science, International Conference, COSIT 2003*, pages 118–135, 2003.

9 Michael Leyton. A process-grammar for shape. *Artificial Intelligence*, 34:213–247, 1988.

10 Gérard Ligozat. *Qualitative Spatial and Temporal Reasoning*. Wiley, 2011.

11 Reinhard Moratz. Representing relative direction as a binary relation of oriented points. In *Proceedings of the 17th Eureopean Conference on Artificial Intelligence, ECAI'2006*, pages 407–411, 2006.

12 Till Mossakowski and Reinhard Moratz. Qualitative reasoning about relative direction of oriented points. *Artificial Intelligence*, 180-181:34–45, 2012.

13 Albert Pich and Zoe Falomir. Logical composition of qualitative shapes applied to solve spatial reasoning tests. *Cognitive Systems Research*, 52:82–102, 2018.

14 Kiyoji Shiono. Development of geology-oriented logical system (geo-logics): Search for mathematical bases of geological data processing (in japanese). *Geoinformatics*, 30(4):161–179, 2019.

15 Michael Sioutis and Diedrich Wolter. Qualitative spatial and temporal reasoning: current status and future challenges. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence*, pages 4594–4601, 2021.

16 Spiros Skiadopoulos and Manolis Koubarakis. Composing cardinal direction relations. *Artificial Intelligence*, 152(2):143–171, 2004.

17 Yuta Taniuchi and Kazuko Takahashi. Spatial representation and reasoning about fold strata: A qualitative approach. In *15th International Conference, ICAART 2023, Revised Selected Papers*, pages 244–266, 2024.

18 Momo Tosue and Kazuko Takahashi. Towards a qualitative reasoning on shape change and object division. In *14th International Conference on Spatial Information Theory (COSIT 2019)*, pages 7:1–7:15, 2019.