# What Does Qualitative Spatial Knowledge Tell About Origami Geometric Folds?

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**Abstract.** Origami geometry is based on a set of 7 fundamental folding operations. By applying a well-chosen sequence of the operations, we are able to solve a variety of geometric problems including those impossible by using Euclidean tools. In this paper, we examine these operations from spatial qualitative point of view, i.e. a common-sense knowledge of the space and the relations between its objects. The qualitative spatial representation of the origami folds is suitable for human cognition when practicing origami by hand. We analyze the spatial relations between the parameters of the folding operations using some existing spatial calculus. We attempt to divide the set of possible values of the parameters into disjoint spatial configurations that correspond to a specific number of fold lines. Our analyses and proofs use the power of a computer algebra system and in particular the Gröbner basis algorithm.

**Keywords:** Origami geometry · Huzita-Justin folding operations
Region Connection Calculus · Relative distance

## 1 Introduction

Origami is the art of paper folding and can serve as framework for solving geometric problems. Seven fundamental operations have been defined by Huzita [8] and Justin [9] to show how to fold the origami and make variety of geometric objects and in particular objects that require solving cubic equations. Origami is simple as only hands are involved in the folding process, affordable as paper is abundant and powerful as it solves problems unsolvable by using straightedge and compass. These advantages give grounds for incorporating origami in a lesson of geometry. Are the fundamental operations of origami geometry suitable for human (or a pupil) cognition?

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Research on the fundamental origami operations focused on their possibilities or increasing their power [1,10,11]. However, anyone who has ever struggled with the challenge of making a geometrical origami object by hand, is familiar with the difficulty of applying the 6th operation. The operation goes as follow (as originally stated by Huzita).

> (6) Given two distinct points and two distinct lines, you can fold superposing the first point onto the first line and the second point onto the second line at the same time.

The 6th operation requires superposing two points on two lines simultaneously. We invite the reader to try it with a piece of paper. Martin advised to use a transparent paper or to fold in front of a lightbulb [12]. Others hinted that the fold includes sliding a point on a line to bring the other point on the other line [14].

A diagram in Euclidean geometry or a shape in origami geometry is, in the first place, a collection of spatial objects such as lines, points, segment lines, circles, etc. The spatial knowledge is given by relations that describe a common sense understanding of the space and its objects. Examples of such relations are on, inside, outside, to the left, to the right, etc. These relations are rudimentary in the sense they can be described by the naked eye without further calculations or reasoning and thus suitable for human cognition. We attempt at providing a qualitative representation of the fundamental fold operations.

In this paper, we build on the first author's previous work [7]. The paper [7] analyzes the fundamental fold operations by identifying the degenerate cases, enumerating the cases where some operations can be derived from others, among other things. The degenerate cases are configurations of points and lines on the origami where the fold operation is not well-defined because of infinite possibilities. By excluding these cases, the fold operation has a finite number of solutions and thus well defined. In this paper, we further develop this analysis. We divide the origami space into disjoint configurations that give an exact number $n$ of fold lines, where $0 \leq n \leq 3$. To that end, we present a mapping of the spatial relations and the fold operations into algebraic terms. We also present a systematic proof strategy to show the statements on the number of fold lines.

The rest of the paper is organized as follows. We first introduce origami geometry based on the fundamental fold operations in Sect. 2. Then, in Sect. 3, we explain the various qualitative calculi that we use. The spatial configuration of well definedness are listed in Sect. 4 and the configurations on the number of solutions are explained in Sects. 5 and 6. Finally, in Sect. 7, we conclude.

## 2 Origami Spatial Objects and Their Construction

### 2.1 Origami Shape

We work with a square origami paper. By hand, we can fold the origami paper and make a crease. A crease leaves a trace on the origami paper, a line segment

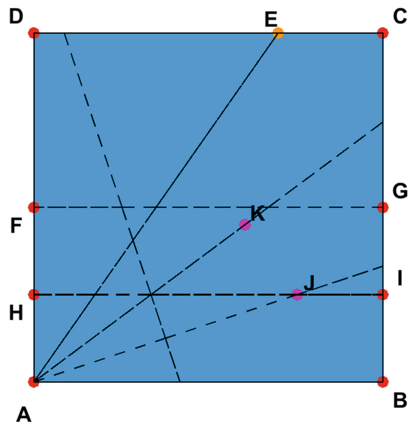



**Fig. 1.** An origami geometric shape: lines $AK$ and $AJ$ are trisectors of $\angle EBA$

whose endpoints are on the edges of the origami square paper. The extension of the line segment of a crease is a line that we call *fold line*. The intersection of two non-parallel fold lines is an origami point that can be outside the origami square.

An origami construction is a sequence of folds (and unfolds). When the collection of points and lines, constructed by folds, have a geometric meaning, we say that we constructed an origami geometric shape. For instance, the origami geometric shape in Fig. 1 depicts two line trisectors $AK$ and $AJ$ of angle $\angle EAB$. The remaining points and line segments on the origami, e.g. $F$, $G$, $H$, $I$, are constructed during the intermediate steps and used to make the trisectors.

### 2.2 Origami Fold

How to obtain a meaningful origami shape such as the one in Fig. 1? We need, foremost, a rigorous definition of the origami folds in the way Euclid's Elements define constructions with a compass and a straightedge. Let $\mathcal{O}$ be an origami square $\square ABCD$. An origami shape is obtained by applying the following fundamental fold operations [8,9].

**(O1)** Given two distinct points $P$ and $Q$, fold $\mathcal{O}$ along the unique line that passes through $P$ and $Q$.
**(O2)** Given two distinct points $P$ and $Q$, fold $\mathcal{O}$ along the unique line to superpose $P$ and $Q$.
**(O3)** Given two distinct lines $m$ and $n$, fold $\mathcal{O}$ along a line to superpose $m$ and $n$.
**(O4)** Given a line $m$ and a point $P$, fold $\mathcal{O}$ along the unique line passing through $P$ to superpose $m$ onto itself.
**(O5)** Given a line $m$, a point $P$ not on $m$ and a point $Q$, fold $\mathcal{O}$ along a line passing through $Q$ to superpose $P$ and $m$.
**(O6)** Given two lines $m$ and $n$, a point $P$ not on $m$ and a point $Q$ not on $n$, where $m$ and $n$ are distinct or $P$ and $Q$ are distinct, fold $\mathcal{O}$ along a line to superpose $P$ and $m$, and $Q$ and $n$.

**(O7)** Given two lines $m$ and $n$ and a point $P$ not on $m$, fold $\mathcal{O}$ along the unique line to superpose $P$ and $m$, and $n$ onto itself.

The fold line described by operation (O1) is the line passing through two distinct points. The fold line described by operation (O2) is the perpendicular bisector of the line segment $PQ$ as shown in Fig. 2. Operation (O3) gives rise to at most two fold lines, which are the interior and exterior bisectors of the angle formed by the two lines $m$ and $n$. To perform operation (O4), we drop a line perpendicular to $m$ and passing through $P$. The fold line of operation (O5) is the line tangent to the parabola of focus $P$ and directrix $m$, denoted by $\mathcal{P}(P, m)$, and passing through point $Q$. This operation is shown in Fig. 3. The operation (O6) in Fig. 4 is about finding a common tangent to the parabolas $\mathcal{P}(P, m)$ and $\mathcal{P}(Q, n)$. Finally, to perform operation (O7), we fold along the tangent to the parabola $\mathcal{P}(P, m)$ and perpendicular to the line $n$.

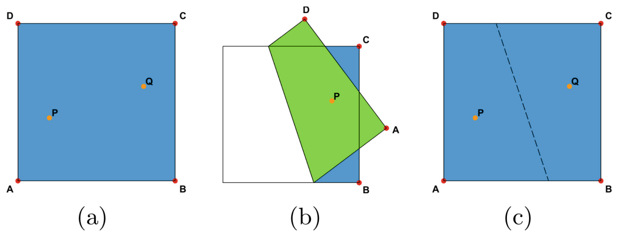



**Fig. 2.** Operation (O2)

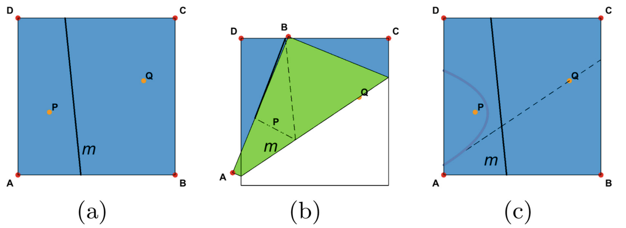



**Fig. 3.** Operation (O5)

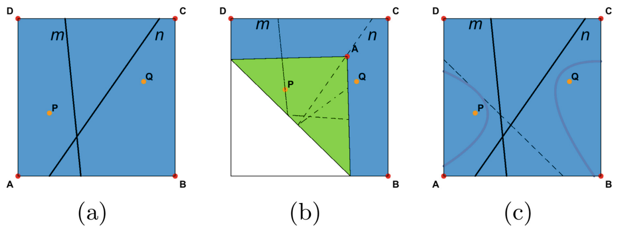



**Fig. 4.** Operation (O6)

## 3 Qualitative Spatial Relations in Origami

The following common sense concepts of connection, orientation and distance will be used to describe origami folds qualitatively.

### 3.1 Object Connection

There is a finite number of situations on the way objects are put together. For instance, whether they are connected and, if true, in which way they are connected. Such situations are described using Region Connection Calculus known in literature by RCC [3]. RCC defines a set of spatial relations. The commonly used ones are either 5 or 8 relations depending on the topology of the spatial objects and the purpose of the representation. Nevertheless, the set of relations must satisfy an important property: pairwise disjoint and jointly exhaustive, which means exactly one relation holds between two arbitrary objects.

RCC5 works for an object equal to its topological closure, in other words its boundary and interior coincide. This is the case of origami points and lines. Table 1 describes all possible connections between points and lines without ambiguities. Note that the 5 spatial relations are equivalent to basic geometric properties in the 2D plane. For instance, the relation `proper-part` stands for the geometric property that a point is on a line, two lines are `disjoint` when they are parallel, etc.

**Table 1.** RCC5 relations between origami points and lines

| | equal | proper-part | intersect | proper-part$^{-1}$ | disjoint |
|---|---|---|---|---|---|
| Point×Point | $P_1, P_2$ | – | – | – | $P_1$ $P_2$ |
| Line×Line | $m, n$ | – | $m$ $n$ | – | $m$ $n$ |
| Point×Line | – | $m$ $P$ | – | – | $m$ $P$ |
| Line×Point | – | – | – | $m$ $P$ | $m$ $P$ |

Circles are more complex objects. The RCC5 is limited since we cannot distinguish between a line tangent to a circle and a line not intersecting a circle, or when two circles are tangent or disjoint. We use two of RCC8 relations, namely relations `disconnected` and `externally-connected`, to improve the expressiveness as shown in Fig. 5 [13].

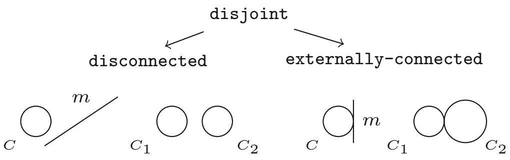



**Fig. 5.** Two relations of RCC8 to describe that circles and lines are disjoint

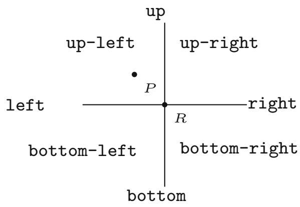



**Fig. 6.** Relations of relative orientation with $(P, R) \in$ up-left

### 3.2 Relative Orientation

A well-known qualitative description of the positions of objects relative to each other is Freska's calculus for points on the 2D plane [6]. To describe the position of an origami point $P$ with respect to a reference point $R$, we divide the origami plane into 8 regions intersecting in $R$ as shown in Fig. 6.

### 3.3 Relative Distance

Several approaches have been defined to compare lengths of intervals which can be regarded as distance between ending points [4]. However, these approaches do not make a good use of the possibilities that the space may offer. The origami plane, for instance, is a dynamic medium. By means of folds, points can be moved by reflection while preserving the length.

**Example.** We want to compare the distances $d(P, Q)$ and $d(R, S)$ in Fig. 7(a), where $d$ is the conventional Euclidean distance. First, we perform an (O3) fold to bring points $R$ and $S$ on the line $PQ$ as shown in Fig. 7(b). In Fig. 7(c), $R1$ and $S1$ are the reflections of $R$ and $S$ by the fold.[1] Next, we perform an (O2) fold to bring $R1$ onto $P$. Figures 7(d) and (e) show this operation, where $R2 = P$ is the reflection of $R1$ and $S2$ the reflection of $S1$. Finally, we perform along the line that passes through $R2$ and perpendicular to line $PQ$, i.e. (O4) fold. The operation is depicted in Fig. 7(f) and (g) shows a new point $S3$ obtained by the reflection of $S2$ by the fold. Since folding preserves the distance, we have

$$d(R, S) = d(R1, S1) = d(R2, S2) = d(P, S2) = d(P, S3).$$

The points $P$, $S3$ and $Q$ are aligned consecutive points in a homogeneous distance system where any given interval is bigger or equal than the previous one [4]. Since $S3$ and $Q$ are disjoint, $d(P, Q) > d(P, S3) = d(R, S)$.

Distance between a point and a line, e.g. $d(P, m)$, is in essence a distance between points. For instance, in the case of $d(P, m)$, we perform (O4) along the point $P$ and perpendicular to $m$. Let $Q$ be the intersection of the fold line and

[1] The reflection point can be easily obtained by (O1)–(O7) folds. We omit the steps in Fig. 7.

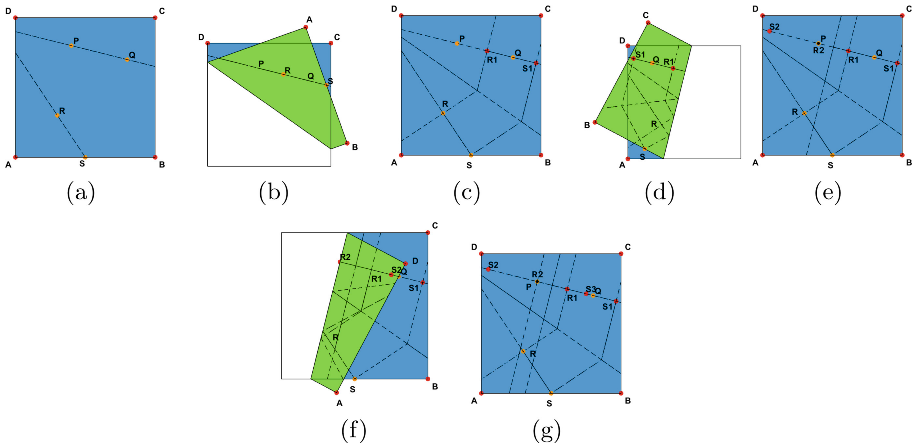



**Fig. 7.** Origami folds to deduce that $d(P, Q) \geq d(R, S)$

$m$. Then, $d(P, m) = d(P, Q)$. Similarly, the distance between two parallel lines, e.g. $d(m, n)$, is defined as the distance between two points $M$ and $N$ on the lines, where $MN \perp m$.

## 4 Well-Definedness of Fold Operations

The statements of (O1)–(O7) in Sect. 2.2 include conditions like "two distinct points $P$ and $Q$", "two distinct lines $m$ and $n$", "a point $P$ not on $m$". These are the conditions to eliminate degenerate configurations or incidence configurations. The degenerate situations are configurations of points and lines where there are infinite possibilities for the fold line. The incidence configurations occurs when we superpose a point $P$ and a line $m$ and $(P, m) \in$ `proper-part`. The operation becomes solvable with simpler operations, i.e. operations that solve lower degree equations. See [7] for a discussion on the configurations of degeneracy and incidence.

These conditions are intuitive and can be expressed qualitatively. First, we use the following lemma. Distinct points (respectively distinct lines) means not equal (respectively lines not equal).

**Lemma 1.** – $(P, Q) \notin$ `equal` *if and only if* $(P, Q) \in$ `disjoint`.
– $(m, n) \notin$ `equal` *if and only if* $(m, n) \in$ `intersects` $\cup$ `disjoint`.
– $(P, m) \notin$ `proper-part` *if and only if* $(P, m) \in$ `disjoint`.

The proof follows from the fact that the RCC5 relations between points and lines are jointly exhaustive and pairwise distinct.

– The Operation (O1) is well defined when $(P, Q) \in$ `disjoint`.
– The Operation (O2) is well defined when $(P, Q) \in$ `disjoint`.

- The Operation (O3) is well defined when $(m,n) \in$ `intersects` $\cup$`disjoint`.
- The Operation (O4) is always well defined.
- The Operation (O5) is well defined when $(P,m) \in$ `disjoint`.
- The Operation (O6) is well defined when $(P,m) \in$ `disjoint` and $(Q,m) \in$ `disjoint` and $((m,n) \in$ `intersects` $\cup$`disjoint` or $(P,Q) \in$ `disjoint`.
- The Operation (O7) is well defined when $(P,m) \in$ `disjoint`.

### 4.1 Spatial Conditions of the Solutions of (O1)–(O4)

Non-degenerate configurations are further processed by identifying the number of solutions. The solutions of operations (O1), (O2), (O3) and (O4) are straightforward. We can easily show that (O1) and (O2) have a unique solution if and only if the points are not equal, whereas (O4) always has a unique solution independently from the spatial configuration of the parameters. Operation (O3) has two solutions if the lines parameters are in relation `intersects` and one solution if the lines are in `disjoint`.

## 5 Spatial Conditions of the Solutions of (O5)–(O7)

### 5.1 A Systematic Approach

**Objects.** To analyze the solutions of (O5)–(O7), we use an algebraic approach. We consider a Cartesian Coordinate system. Points are defined by pairs of their coordinates. We denote the coordinates of a point $P$ by $(x_p, y_p)$. A well defined line has an equation of the form $ax + by + c = 0$, where $a \neq 0 \vee b \neq 0$. We denote by $a_m, b_m$ and $c_m$ the coefficients of a line $m$. A circle $\mathcal{C}(P,r)$, whose center and radius are $P$ and $r > 0$, has the equation $\sqrt{(x - x_p)^2 + (y - y_p)^2} = r$.

The determination of the (exact) domain of the coordinates and coefficients is tricky. $\mathbb{Q}$ is too small since it doesn't include $\sqrt{x}$ numbers and $\mathbb{R}$ is too much. An algebraic extension of $\mathbb{Q}$ would be a good candidate since the origami fundamental fold operations allow the construction of rational numbers plus numbers of the form $\sqrt{x}$ and $\sqrt[3]{x}$ [5].

**Algebraic Relations and Functions.** Table 2 shows the algebraic relations of the qualitative spatial relations explained in Sect. 3. The algebraic forms are self-explanatory.

Furthermore, in our analysis of the fold operations, specifically operations (O5)–(O7), we work with parabolas $\mathcal{P}(P,m)$ represented by the following equation $f(x,y)$.

$$f(x,y) := (x - x_p)^2 + (y - y_p)^2 - \frac{(a_m x + b_m y + c_m)^2}{{a_m}^2 + {b_m}^2} = 0 \tag{1}$$

Let $t$ be a tangent to the parabola $\mathcal{P}(P,m)$ at a point $(x_1, y_1)$. Also, let $\lambda$ be the slope of $t$. Then the following equation $g(x_1, y_1)$ defines the tangent $t$.

$$g(x_1, y_1) := \frac{\partial f}{\partial x}(x_1, y_1) + (\frac{\partial f}{\partial y}(x_1, y_1))\lambda = 0 \tag{2}$$

**Table 2.** Algebraic forms of the qualitative spatial relations and functions

| | Spatial relation/function | Algebraic relation/function |
|---|---|---|
| Object connection | $(P, Q) \in$ `equal` | $x_p = x_q \wedge y_p = y_q$ |
| | $(P, Q) \in$ `disjoint` | $x_p \neq x_q \wedge y_p \neq y_q$ |
| | $(m, n) \in$ `equal` | $\exists k.\ a_n = ka_m \wedge b_n = kb_m \wedge c_n = kc_m$ |
| | $(m, n) \in$ `disjoint` | $(a_n b_m - a_m b_n = 0) \wedge \neg((m, n) \in$ `equal`$)$ |
| | $(m, n) \in$ `intersects` | $a_n b_m - a_m b_n \neq 0$ |
| | $(\mathcal{C}(P, r_1), \mathcal{C}(Q, r_2)) \in$ `equal` | $(P, Q) \in$ `equal` $\wedge\, r_1 = r_2$ |
| | $(\mathcal{C}(P, r_1), \mathcal{C}(Q, r_2)) \in$ `intersects` | $((P, Q) \in$ `disjoint`$) \wedge (\sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} < r_1 + r_2)$ |
| | $(\mathcal{C}(P, r_1), \mathcal{C}(Q, r_2)) \in$ `externally-connected` | $((P, Q) \in$ `disjoint`$) \wedge (\sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} = r_1 + r_2)$ |
| | $(\mathcal{C}(P, r_1), \mathcal{C}(Q, r_2)) \in$ `disconnected` | $((P, Q) \in$ `disjoint`$) \wedge (\sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} > r_1 + r_2)$ |
| | $(Q, \mathcal{C}(P, r)) \in$ `proper-part` | $\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} = r$ |
| | $(Q, \mathcal{C}(P, r)) \in$ `disjoint` | $(\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} < r) \vee (\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} > r)$ |
| | $(m, \mathcal{C}(P, r)) \in$ `intersects` | $\frac{\lvert a_m x_p + b_m y_p + c_m \rvert}{\sqrt{a_m^2 + b_m^2}} < r$ |
| | $(m, \mathcal{C}(P, r)) \in$ `externally-connected` | $\frac{\lvert a_m x_p + b_m y_p + c_m \rvert}{\sqrt{a_m^2 + b_m^2}} = r$ |
| | $(m, \mathcal{C}(P, r)) \in$ `disconnected` | $\frac{\lvert a_m x_p + b_m y_p + c_m \rvert}{\sqrt{a_m^2 + b_m^2}} > r$ |
| Orientation | $(P, Q) \in$ `left` | $x_p < x_q \wedge y_p = y_q$ |
| | $(P, Q) \in$ `right` | $x_p > x_q \wedge y_p = y_q$ |
| | $(P, Q) \in$ `up` | $x_p = x_q \wedge y_p > y_q$ |
| | $(P, Q) \in$ `bottom` | $x_p = x_q \wedge y_p < y_q$ |
| | $(P, Q) \in$ `up-left` | $x_p < x_q \wedge y_p > y_q$ |
| | $(P, Q) \in$ `up-right` | $x_p > x_q \wedge y_p > y_q$ |
| | $(P, Q) \in$ `bottom-left` | $x_p < x_q \wedge y_p < y_q$ |
| | $(P, Q) \in$ `bottom-right` | $x_p > x_q \wedge y_p < y_q$ |
| Distance | $d(P, Q)$ | $\sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$ |
| | $d(P, m)$ | $\frac{\lvert a_m x_p + b_m y_p + c_m \rvert}{\sqrt{a_m^2 + b_m^2}}$ |
| | $d(m, n)$, where $m \parallel n$ | $\frac{\lvert c_m - c_n \rvert}{\sqrt{a_m^2 + b_m^2}}$ |

**Proof Strategy.** To prove the number of the solutions of (O5)–(O7), we perform the following steps.

1. Define a system $S$ of the algebraic relations that describe the fold line. These relations are (1) and (2) as well as well-established algebraic form of geometric properties that we will explain when used.
2. Compute Gröbner basis of $S$. This step attempts to eliminate some of the variables and obtain one equation in the slope of the fold line.
   - If one polynomial is obtained, then compute its discriminant.
   - If more than one polynomial are obtained, then solve for some of the dependent variables.
3. Analyze the obtained polynomial expressions to identify the cases with real solutions. Specifically, we watch out for the appearance of relations of Table 2.

We use the power of the computer algebra system *Mathematica* to perform the computations in the above steps. We use Buchberger's algorithm to generate Gröbner bases. The computations are performed symbolically, thus we prove our results in the general case.

### 5.2 Spatial Conditions of the Solutions of (O5)

**Theorem 2.** *Let $P$, $Q$ and $m$ be on the origami where $(P, m) \in$ `disjoint`. We perform the (O5) operation along the fold line passing through $Q$ to superpose $P$ and $m$.*

- *If d(Q, P) = d(Q, m), then there is a unique fold line.*
- *If d(Q, P) > d(Q, m), then there are two distinct fold lines.*
- *If d(Q, P) < d(Q, m), then there is no fold line.*

*Proof.* Since $(P, m) \in$ `disjoint`, we know that the solutions of (O5) are the lines passing through $Q$ and tangent to the parabola $\mathcal{P}(P, m)$. Let $\lambda$ be the slope of the tangent to the parabola $\mathcal{P}(P, m)$ at point $(x_1, y_1)$. Hence, we have the system of equations $S = \{f(x_1, y_1),\ g(x_1, y_1), (y_1 - y_q) - \lambda(x_1 - x_q) = 0\}$. $f$ and $g$ are given in (1) and (2), respectively. The equation $(y_1 - y_q) - \lambda(x_1 - x_q) = 0$ means that the tangent passes through the points $(x_1, y_1)$ and $Q$.

We compute the Gröbner basis of $S$. We obtain a 2nd degree polynomial in $\lambda$ whose discriminant is the following.

$$4(a_m^2 + b_m^2)^2(c_m + a_m x_p + b_m y_p)^2 \times \tag{3}$$

$$(a_m^2 + b_m^2)((x_p - x_q)^2 + (y_p - y_q)^2) - (a_m x_q + b_m y_q + c_m)^2 \tag{4}$$

Line $m$ is well defined, then $a_m$ and $b_m$ cannot vanish at the same time and $a_m^2 + b_m^2 > 0$. Also, since $(P, m) \in$ `disjoint`, $(c_m + a_m x_p + b_m y_p)^2 > 0$. Thus, the factors in (3) are always strictly positive and the sign of the polynomial in (4) determines the number of solutions of $\lambda$. (4) is the expression of $d(Q, P)^2 - d(Q, m)^2$ in algebraic terms. If strictly positive, we have two solutions for $\lambda$, i.e. two fold lines. If strictly negative, then there is no solution. If equal to 0 then there is a unique fold line. □

From a geometric point of view, a tangent $t$ to a parabola is the perpendicular bisector of the line segment joining $P$ and a point on $m$ that we name $P'$. Since $t$ passes through $Q$, the circle whose center is $Q$ and radius $\overline{QP}$ intersects $m$ in $P'$. In Fig. 8, we show the situation where the circle intersects $m$ in two points and we have two tangents or two fold lines $t_1$ and $t_2$. If the circle does not intersect $m$, then no tangent exists. If the circle is tangent to $m$ ($Q$ is a point on the parabola) then there is one tangent.

### 5.3 Spatial Conditions of the Solutions of (O7)

**Theorem 3.** *Let $P$, $m$ and $n$ be on the origami where $(P, m) \in$ `disjoint`. We perform operation (O7) along the fold line $t$ perpendicular to $n$ to superpose $P$ and $m$.*

- *If $(m, n) \in$ `equal` $\cup$ `disjoint`, then there is no fold line.*
- *If $(m, n) \in$ `intersects`, then there is one fold line.*

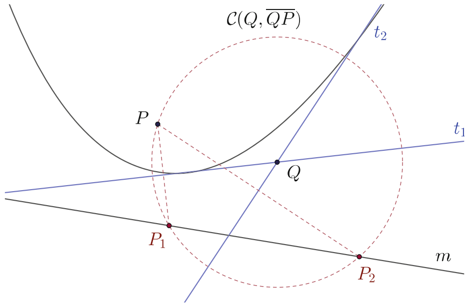



**Fig. 8.** $d(Q,P) > d(Q,m)$: two fold lines $t_1$ and $t_2$ for (O5)

*Proof.* We solve for $x_1$ and $y_1$ in $\{f(x_1, y_1),\ g(x_1, y_1),\ a_n\lambda - b_n = 0\}$, where $\lambda$ is the slope of the fold line, and $f$ and $g$ are defined in (1) and (2). The former equation $a_n\lambda - b_n = 0$ states that $n$ and the tangent are perpendicular. We obtain one solution of the following form:

$$x_1 \to \frac{\text{some polynomial expression of } a_m, b_m, c_m, a_n, b_n, c_n, x_p \text{ and } y_p}{2(a_n b_m - a_m b_n)^2}$$

$$y_1 \to \frac{\text{some polynomial expression of } a_m, b_m, c_m, a_n, b_n, c_n, x_p \text{ and } y_p}{2(a_n b_m - a_m b_n)^2}$$

However, the solutions are undefined when the denominator is null, i.e. $a_n b_m - a_m b_n = 0$. This is the algebraic relations of two lines that are `disjoint` or `equal` according to Table 2. □

### 5.4 Spatial Conditions of the Solutions of (O6) with Disjoint Lines

**Theorem 4.** *Let points $P$ and $Q$ and* `disjoint` *lines $m$ and $n$ be on origami, where $(P, m) \in$* `disjoint` *and $(Q, m) \in$* `disjoint`*. We perform operation (O6) to superpose $P$ and $m$ and $Q$ and $n$.*

- *If $d(m, n) > d(P, Q)$, then there is no fold line.*
- *If $d(m, n) = d(P, Q)$, then there is a unique fold line.*
- *If $d(m, n) < d(P, Q)$, then there are two fold lines.*

*Proof.* We proceed similarly to the proof of Theorem 2. We know that the fold line is a common tangent to parabolas $\mathcal{P}(P, m)$ and $\mathcal{P}(Q, n)$. We compute the Gröbner basis of $\{f_1(x_1, y_1), g_1(x_1, y_1), f_2(x_2, y_2), g_2(x_2, y_2), (y_1 - y_2) - (x_1 - x_2)\lambda = 0, a_n b_m - a_m b_n = 0\}$, where $\lambda$ is the slope of the common tangent of $\mathcal{P}(P, m)$ and $\mathcal{P}(Q, n)$ at $(x_1, y_1)$ and $(x_2, y_2)$, respectively. Note that $f_1$, $g_1$, $f_2$ and $g_2$ are Eqs. (1) and (2) defined for the first parabola $\mathcal{P}(P, m)$ and the second parabola $\mathcal{P}(Q, n)$. The discriminant of the result of Gröbner basis computation gives

$$(a_m^2 + b_m^2)((x_p - x_q)^2 + (y_p - y_q)^2 - (c_m - c_n)^2),$$

which stands for the algebraic form of $d(P, Q)^2 - d(m, n)^2$. □

# 6 Analysis of (O6) with Intersecting Lines

## 6.1 Simplification Without Loss of Generality

Operation (O6) contributes to geometry by solving problems that are impossible by classical Euclidean tools. The operation has the merits of solving any cubic equation of the general form $ax^3 + bx^2 + cx + d = 0$. Coefficients $a$, $b$, $c$ and $d$ are in the field of origami constructible numbers, i.e. an algebraic extension of $\mathbb{Q}$ with square root and cubic square root [5].

To simplify our analysis of operation (O6), we use lines parallel to $xy$-axes. This reduces the number of parameters that come from the lines $m$ and $n$ and simplifies the Gröbner basis computation, which is, in the worst case, double exponential in the number of variables [2].

**Lemma 5.** *Any cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, can be solved with lines $m$ and $n$ perpendicular and parallel to $xy$-axes, respectively.*

*Proof.* We apply (O6) to superpose $P$ and $m$, and $Q$ and $n$, simultaneously. The fold line is a common tangent to the parabolas $\mathcal{P}(P, m)$ and $\mathcal{P}(Q, n)$. Let $\lambda$ be the slope of the common tangent. We take $m$ and $n$ to be of equations $x + c_m = 0$ and $y + c_n = 0$. We compute the Gröbner basis of

$$\{f_1(x_1, y_1), f_2(x_2, y_2), g_1(x_1, y_1), g_2(x_2, y_2), (y_2 - y_1) - \lambda(x_2 - x_1) = 0\},$$

where the former equation $(y_2 - y_1) - \lambda(x_2 - x_1) = 0$ states that the tangent passes through the points $(x_1, y_1)$ and $(x_2, y_2)$ on the parabolas. The result is a cubic polynomial in $\lambda$.

$$(c_n + y_q)\lambda^3 + (c_m - x_p + 2x_q)\lambda^2 + (c_n + 2y_p - x_q)\lambda + c_m + x_p \qquad (5)$$

We match the coefficient of the above polynomial with $a$, $b$, $c$ and $d$. We solve for the coordinates of $P$ and $Q$ and obtain:

$$\{x_p \to -c_m + d, y_p \to (a + c - 2c_n)/2, x_q \to (b - 2c_m + d)/2, y_q \to a - c_n\} \quad (6)$$

□

We can further simplify by taking $c_m = c_n = 0$. In that case, solutions in (6) gives rise to $P(d, (a + c)/2)$ and $Q((b + d)/2, a)$. For instance, to solve the cubic $x^3 - 3x^2 + \frac{27}{8} = 0$ with (O6), we can take the lines $m : x = 0$ and $n : y = 0$ and the points $P(\frac{27}{8}, \frac{1}{2})$ and $Q(\frac{3}{16}, 1)$.

## 6.2 The Discriminant Function

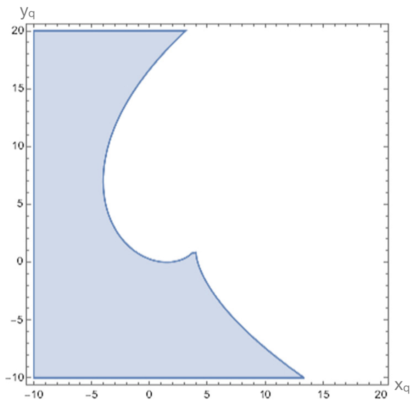
**Lemma 6.** *Let $\Delta$ be the discriminant of the polynomial (5). We have the following result about the number of real roots.*

*(i) If $\Delta < 0$, then polynomial (5) has a single real root.*
*(ii) If $\Delta > 0$, then polynomial (5) has three distinct real roots.*

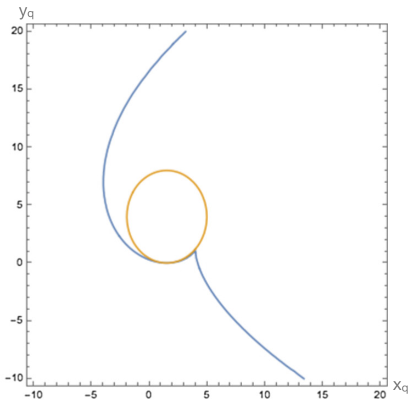
*(iii) If $\Delta = 0$, then polynomial (5) has either a triple real root, or one double real root and one single real root.*

*Proof.* The proof is a result of solving cubic equations with radicals. □

The description of $\Delta$ in term of spatial relations between (O6) parameters $m$, $n$, $P$ and $Q$ is not straightforward. So, we observe how point $Q$ would relate to $P$. We fix $P$ to be the point $(3, 4)$, for instance, and take $m$ and $n$ to be the $y$-axis and the $x$-axis, respectively. We plot $\Delta(x_q, y_q) = 0$. Figure 9(a) depicts the 3 regions defined by the curve $\Delta(x_q, y_q) = 0$. If $Q$ is on the blue region then we are in case (ii), i.e. there are 3 distinct fold lines, if on the white region then case (i), i.e. one fold line, if on the curve then case (iii), i.e. either one or two fold lines.




(a)




(b)

**Fig. 9.** The curve $\Delta(x_q, y_q) = 0$ (Color figure online)

Hereafter, we give a geometric explanation of the curve $\Delta(x_q, y_q)$. In Fig. 10, we take two parabolas tangent at point $S$. Obviously, a possible fold line is the tangent passing through point $S$. We move the point $S$ on the parabola $\mathcal{P}(P, m)$ and trace the point $Q$. The locus of point $Q$ is $\Delta(x_q, y_q) = 0$. Since the parabolas are tangent, then 2 or 3 fold lines coincide and thus correspond to double or triple real roots.

### 6.3 Spatial Case 1: $Q$ is Equal to the Cusp Point

**Lemma 7.** *If $Q$ is the cusp point of $\Delta(x_q, y_q) = 0$, then (O6) has a unique fold line.*

*Proof.* Another useful constant of a cubic $ax^3 + bx^2 + cx + d = 0$ is $\Delta_0 = b^2 - 3ac$. When $\Delta_0 = \Delta = 0$, there exists a triple real root. In our example $\Delta_0(x_q, y_q) = 0$

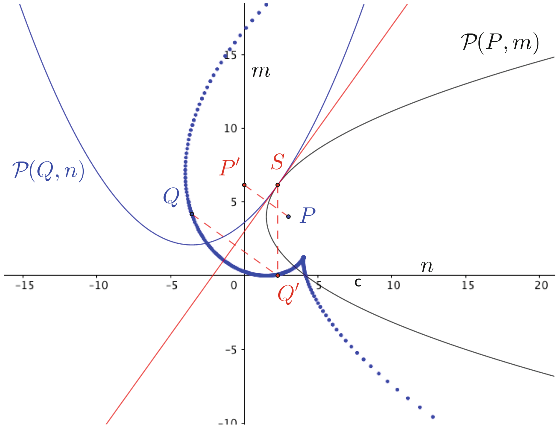



**Fig. 10.** The locus of $Q$ (blue curve) when $P$, $m$ and $n$ are fixed and $\mathcal{P}(P,m)$, and $\mathcal{P}(Q,m)$ are tangent at a point $S$ (Color figure online)

is shown in Fig. 9(b). The cusp point is an intersection point of the two curves and correspond to a situation where we have one fold line of multiplicity 3. A second intersection point is on $n$ excluded by the condition $(Q,n) \in$ `disjoint` of (O6). □

Figure 11 shows the circles when $Q$ is the cusp point. We have the following result based on spatial observation.

**Lemma 8.** *Let point $O$ be the intersection of $m$ and $n$, and point $M$ be the middle point of the line segment $PQ$. Furthermore, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the circles $\mathcal{C}(M,\overline{MP})$ and $\mathcal{C}(O,\overline{PQ})$. If $(\mathcal{C}_1,\mathcal{C}_2) \in$ `externally-connected`, then $Q$ is the cusp point.*

*Proof.* We provide the sketch of the proof. Using *Mathematica*:

1. Solve for the coordinates $x_q$ and $y_q$ of the cusp point using

$$\frac{\partial \Delta(x_q,y_q)}{\partial y_q} = \frac{\partial \Delta(x_q,y_q)}{\partial x_q} = 0.$$

2. Show that the circles $\mathcal{C}(M,\overline{MP})$ and $\mathcal{C}(O,\overline{PQ})$ are `externally-connected`using the appropriate relation from Table 2.

□

### 6.4 Spatial Case 2: *P* and *Q* are on Opposite Half-Planes

The origami plane is divided by lines $m$ and $n$ into the 8 regions of relative orientation (see Sect. 3.2). The curve $\Delta(x_q,y_q) = 0$ intersects only 3 half-planes.

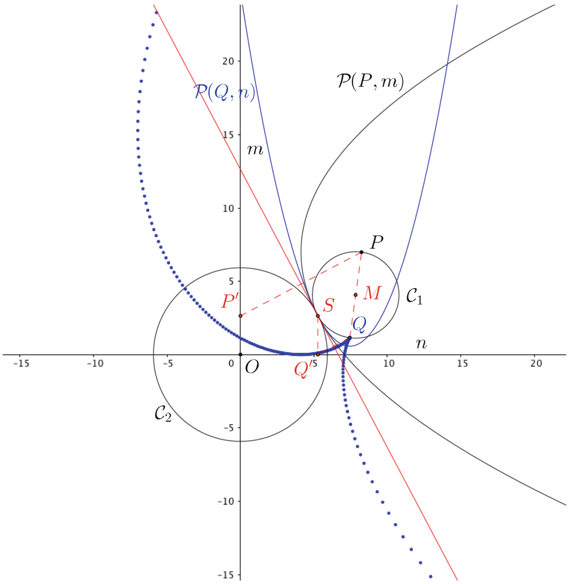



**Fig. 11.** $(\mathcal{C}_1, \mathcal{C}_2) \in$ `externally-connected` when $Q$ is the cusp of $\Delta(x_q, y_q) = 0$

Referring to Fig. 10, for instance, the curve $\Delta(x_q, y_q) = 0$ intersects the `up-left`, `up-right` and `bottom-right` half-planes. The remaining `bottom-left` half-plane is a subset of the region $\Delta(x_q, y_q) > 0$. Therefore, if we take $Q$ to be any point on the `bottom-left` half-plane, then we have 3 distinct fold lines for (O6).

## 7 Conclusion

We analyzed the fundamental folding operations. Based on Gröbner basis and other computer algebra methods, we proved the conditions on the number of fold lines. The conditions are described using qualitative relations between points and lines parameters of the fold operations. This approach worked for operations (O1)–(O5), (O7) and (O6) with disjoint lines. In the case of (O6) with intersecting lines, the spatial configurations cannot be described in a simple qualitative language. To tackle this operation, we identified some spatial cases that are easy to recognize when performing origami by hands.

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