Bandwidth Consecutive Multicolorings of Graphs

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Abstract

Let $G$ be a simple graph in which each vertex $v$ has a positive integer weight $b(v)$ and each edge $(v, w)$ has a nonnegative integer weight $b(v, w)$. A bandwidth consecutive multicoloring of $G$ assigns each vertex $v$ a specified number $b(v)$ of consecutive positive integers so that, for each edge $(v, w)$, all integers assigned to vertex $v$ differ from all integers assigned to vertex $w$ by more than $b(v, w)$. The maximum integer assigned to a vertex is called the span of the coloring. In this thesis, we first investigate fundamental properties of such a coloring. We then obtain a pseudo polynomial-time exact algorithm and a fully polynomial-time approximation scheme for the problem of finding such a coloring of a given series-parallel graph with the minimum span. We finally extend the results to the case where a given graph $G$ is a partial $k$-tree, that is, $G$ has a bounded tree-width.

Summary

An ordinary coloring of a graph $G$ assigns each vertex a color so that, for each edge $(v, w)$, the color assigned to $v$ differs from the color assigned to $w$ [3]. The problem of finding a coloring of a graph $G$ with the minimum number $\chi(G)$ of colors often appears in the scheduling, task-allocation, etc. However, it is NP-hard, and difficult to find a good approximate solution. More precisely, for all $\varepsilon > 0$, approximating $\chi(G)$ within $n^{1-\varepsilon}$ is NP-hard, where $n$ is the number of vertices in $G$.

The ordinary coloring has been extended in various ways [1, 2, 3, 4, 5]. A multicoloring assigns each vertex a specified number of colors so that, for each edge $(v, w)$, the set of colors assigned to $v$ is disjoint with the set of colors assigned to $w$ [1, 2, 5]. A bandwidth coloring assigns each vertex a positive integer as a color so that the two integers assigned to the ends of each edge $(v, w)$ differ by at least the specified weight $\omega(v, w)$ of $(v, w)$ [4].

In this thesis we deal with another generalized coloring, called a “bandwidth consecutive multicoloring.” Let $G = (V, E)$ be a simple graph with vertex set $V$ and edge set $E$. Each vertex $v \in V$ has a positive integer weight $b(v)$, while each edge $(v, w) \in E$ has a non-negative integer weight $b(v, w)$. A bandwidth consecutive multicoloring $F$ of $G$ is an assignment of positive integers to vertices such that

(a) each vertex $v \in V$ is assigned a set $F(v)$ of $b(v)$ consecutive positive integers; and

(b) for each edge $(v, w) \in E$, all integers assigned to $v$ differ from all integers assigned to vertex $w$ by more than $b(v, w)$.

We call such a bandwidth consecutive multicoloring $F$ simply a $b$-coloring of $G$ for a weight function $b$. The maximum integer assigned to a vertex is called the span of a $b$-coloring $F$, and is denoted by $\text{span}(F)$. We define the $b$-chromatic number $\chi_b(G)$ of a graph $G$ to be the minimum span over all $b$-colorings $F$ of $G$. A $b$-coloring $F$ is called optimal if $\text{span}(F) = \chi_b(G)$. A $b$-coloring problem is to compute $\chi_b(G)$ for a given graph $G$.

Figure 1(a) depicts a weighted graph $G$ together with an optimal $b$-coloring $F$ of $G$, where a weight $b(e)$ is attached to an edge $e$, a weight $b(v)$ is written in a circle representing a vertex $v$, and a set $F(v)$ is attached to a vertex $v$. Since $\text{span}(F) = 11$, $\chi_b(G) = 11$.

The ordinary vertex-coloring is merely a $b$-coloring for the case $b(v) = 1$ for every vertex $v$ and $b(v, w) = 0$ for every edge $(v, w)$. The “bandwidth coloring” or “channel assignment” [4] is a $b$-coloring for the case $b(v) = 1$ for every vertex.
v and $b(v, w) = \omega(v, w) - 1$ for every edge $(v, w)$. It should be noted that our edge weight $b(v, w)$ is one less than the ordinary edge weight $\omega(v, w)$ of a bandwidth coloring.

A $b$-coloring arises in the assignment of radio channels in cellular communication systems [4] and in the non-preemptive task scheduling. The $b(v)$ consecutive integers assigned to a vertex $v$ correspond to the contiguous bandwidth of a channel $v$ or the consecutive time periods of a task $v$. The weight $b(v, w)$ assigned to edge $(v, w)$ represents the requirement that the frequency band or time period of $v$ must differ from that of $w$ by more than $b(v, w)$. The span of a $b$-coloring corresponds to the minimum total bandwidth or the minimum makespan.

One can find a multicoloring of a graph $G$ with the minimum number of colors in time polynomial in the output size if $G$ is a series-parallel graph or a partial $k$-tree, that is, a graph of bounded treewidth [2, 5]. The problem of finding a bandwidth coloring with the minimum number of colors is NP-hard even for partial 3-trees [4], and there is a fully polynomial-time approximation scheme (FPTAS) for the problem on partial $k$-trees [4]. Since our $b$-coloring problem is also NP-hard for partial 3-trees, it is desirable to obtain a good approximation algorithm. However, there are only heuristics for the $b$-coloring problem so far.

In this thesis, we first investigate fundamental properties of a $b$-coloring. In particular, we characterize the $b$-chromatic number $\chi_b(G)$ of a graph $G$ in terms of the longest path in acyclic orientations of $G$. We then obtain a pseudo polynomial-time exact algorithm for the $b$-coloring problem on series-parallel graphs, which often appear in the task scheduling and electrical circuits. The algorithm takes time $O(B^3n)$, where $B$ is the maximum weight of $G$: $B = \max_{x \in V \cup E} b(x)$. Using the algorithm, we then give a fully polynomial-time approximation scheme (FPTAS) for the problem. We finally extend these results to the case where $G$ is a partial $k$-tree.

References


Publication